THE QUARTERLY JOURNAL OF

MATHEMATICS

OXFORD SECOND SERIES

Volume 3 *No.* 10 *June* 1952

OXFORD

AT THE CLARENDON PRESS

Price 12*s*. 6*d*. *net*

PRINTED IN GREAT BRITAIN BY CHARLES BATEY AT THE UNIVERSITY PRESS, OXFORD

THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SECOND SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, J. H. C. THOMPSON

THE QUARTERLY JOURNAL OF MATHEMATICS (OXFORD SECOND SERIES) is published at 12s. 6d. net for a single number with an annual subscription (for four numbers) of 44s. 6d. post free.

Papers, of a length normally not exceeding 20 printed pages of the Journal, are invited on subjects of Pure and Applied Mathematics, and should be addressed 'The Editors, Quarterly Journal of Mathematics, Clarendon Press, Oxford'. They should be in a condition fit for sending to a referee and in form and notation convenient for printing. The Editors as a rule will not wish to accept material that they cannot see their way to publish within a twelvemonth.

While every care is taken of manuscripts submitted for publication, the Publisher and the Editors cannot hold themselves responsible for any loss or damage. Authors are advised to retain a copy of anything they may send for publication. Authors of papers printed in the Quarterly Journal will be entitled to 50 free offprints.

Correspondence on the *subject-matter* of the Quarterly Journal should be addressed, as above, to 'The Editors', at the Clarendon Press. All other correspondence should be addressed to the Publisher

GEOFFREY CUMBERLEGE

OXFORD UNIVERSITY PRESS

AMEN HOUSE, LONDON, E.C. 4

THE ROOTS OF THE EQUATION $x = (c\exp)^n x$ AND THE CYCLES OF THE SUBSTITUTION $(x|ce^x)$

By N. D. HAYES (*Aberdeen*)

[Received 8 January 1951]

1. Introduction

THE problem of finding the limiting value of $a^{a^{a^{\cdot^{\cdot^{\cdot}}}}}$ as the number of a increases without limit has led a number of authors to a study of the roots of the equation

$$x = a^x \tag{1.1}$$

for real positive a. Euler (2) in 1778 found the real roots of the equation and gave some information about the imaginary roots. He showed also that for certain values of a the equation

$$x = a^{a^x} \tag{1.2}$$

has real roots other than those of (1.1). Some of these results are also contained in a paper published by Eisenstein (1) in 1844. The most complete investigations of the problem, however, are those of Lémeray (5) into the real and imaginary roots of equation (1.1) and Gravé (3) into the real roots of equations (1.1) and (1.2). I here consider first an equation equivalent to (1.1), namely

$$x = ce^x, \tag{1.3}$$

the behaviour of the roots of which is important in the theory of certain difference-differential equations [see (4)] and in the problem of the iteration of the exponential function [see, for example, (6)]. Closely related to the latter is the study of the cycles of the substitution $(x|ce^x)$ and the roots of the equation

$$x = (c\exp)^n x \tag{1.4}$$

which is my main concern in this paper. Here and elsewhere $(c\exp)^n$ is written for the operator $\{c\exp(c\exp...)\}$, so that, for instance, $(c\exp)^2 x$ means $c\exp(ce^x)$.

In the applications mentioned above the constant c is in general real, and the main theorem of this paper is concerned with this case. The location of the roots of (1.3) and (1.4) when c is complex is, however, also considered and the results given in Theorems 2 and 3 at the end of the paper.

2. Notation

I use the notation $\log_k Q$ (k an integer) to indicate that the value of the logarithm of Q is to be chosen so that

$$2k\pi < \operatorname{im}(\log_k Q) < 2(k+1)\pi.$$

If Q is real and positive and $k \geqslant 0$, then $\operatorname{im}(\log_k Q) = 2k\pi$, while, if $k < 0$, then $\operatorname{im}(\log_k Q) = 2(k+1)\pi$. (Hence in particular, if $Q > 0$, then $\log_0 Q$ and $\log_{-1} Q$ are both real.)

Also, I shall refer to the strip

$$2h\pi < \operatorname{im}(x) < 2(h+1)\pi$$

as 'the strip h'.

3. Principal result

The principal result I shall establish is contained in the following theorem.

THEOREM 1. *Except in the cases mentioned below, there exist, corresponding to each set of positive or negative integers or zeros* $(h_1, h_2,..., h_n)$, *a unique root* x_n $(2h_n\pi < \operatorname{im}(x_n) < 2(h_n+1)\pi)$ *of the equation*

$$x = (c\exp)^n x$$

and a unique set of points $(x_1, x_2,..., x_n)$ $(2h_r\pi < \operatorname{im}(x_r) < 2(h_r+1)\pi)$ *forming a cycle of the n-th order of the substitution* $(x|ce^x)$.

The only exceptions and only additional roots are

(i) *when* $c < 0$, *the equation has a negative real root;*

(ii) *when* $c \leqslant -e$ *and n is even, the equation has two additional negative roots, and there is no complex root corresponding to the set* $(0, -1, 0,..., -1)$ *or to* $(-1, 0, -1,..., 0)$;

(iii) *when* $0 < c < e^{-1}$, *the equation has two real roots and there is no complex root corresponding to the set* $(0, 0, 0,..., 0)$ *or to* $(-1, -1, -1,..., -1)$.

4. Proof of Theorem 1

The proof of Theorem 1 depends directly on Lemmas 5–10, the proofs of which involve the results of Lemmas 1–4 concerning the location of the roots of the equations (1.3) and

$$x = (c\exp)^2 x. \tag{4.1}$$

All the results required concerning the location of the roots of equation (1.3) when c is real are contained in Lemma 1 of my paper (**4**), which for ease of reference I here restate without proof.

LEMMA 1. *The complex roots in the upper half-plane of the equation* $x = ce^x$ *lie, when* $c > 0$, *one in each strip*

$$2p\pi < v < (2p+1)\pi \quad (p = 0, 1, 2,...; \; x = u+iv)$$

at the only intersection of the appropriate branches of the curves

$$v = \pm(c^2e^{2u}-u^2)^{\frac{1}{2}} \tag{4.2}$$

and

$$u = v\cot v \tag{4.3}$$

except that, when $0 < c \leqslant e^{-1}$, *the strip corresponding to* $p = 0$ *contains no intersection and no root.*

When $c < 0$, *the roots lie one in each strip*

$$(2p+1)\pi < v < 2(p+1)\pi \quad (p = 0, 1, 2,...)$$

at the intersection of the appropriate branches of (4.2) *and* (4.3).

In both cases there are corresponding roots in the lower half-plane.

The only real roots of $x = ce^x$ *are, when* $c = e^{-1}$, *a double root* $x = 1$; *when* $0 < c < e^{-1}$, *two roots* ξ_1, ξ_2 $(0 < \xi_1 < 1 < \xi_2)$, *one at each intersection of* (4.2) *with the positive real axis; and, when* $c < 0$, *one root* ξ_3 *at the only intersection of* (4.2) *with the negative real axis.*

A sketch of the curves (4.2) and (4.3) indicating the positions of the roots of $x = ce^x$ for certain specified values of c is given in (4).

The next three lemmas are concerned with the roots of equation (4.1). Lemma 2 is an immediate consequence of the results given by Gravé (3) for the equation $x = a^{a^x}$.

LEMMA 2. *When* $-e < c < 0$, *the only real root of the equation*

$$x = (c\exp)^2x \tag{4.4}$$

is the negative real root ξ_3 *of* $x = ce^x$. *When* $c \leqslant -e$, *the equation has this root and just two additional real roots* λ_1 *and* λ_2 $(c < \lambda_1 \leqslant \xi_3 \leqslant \lambda_2 < 0)$ *such that* $\lambda_1 = ce^{\lambda_2}$ *and* $\lambda_2 = ce^{\lambda_1}$.

Gravé shows also that, if x is real, $c < -e$, and

$$x_1 = ce^x, \quad ..., \quad x_n = ce^{x_{n-1}}, \quad ...,$$

then, for $x < \xi_3$, $x_{2n} \to \lambda_1$ and $x_{2n+1} \to \lambda_2$ as $n \to \infty$, while, for $x > \xi_3$, $x_{2n} \to \lambda_2$ and $x_{2n+1} \to \lambda_1$ as $n \to \infty$. Similarly, when $-e \leqslant c < 0$, the points tend to ξ_3. We need, however, the following more general result.

LEMMA 3. *If* $c < -e$, *then* λ_1 *and* λ_2 *are attractive double points and* ξ_3 *is a repulsive double point of the substitution* $x|(c\exp)^2x$. *If* $-e < c < 0$, *then* ξ_3 *is an attractive double point of the substitution.*

If $\gamma(x) = (c\exp)^2x$, it is easily shown that, for $c < -e$, $\gamma'(\xi_3) > 1$ and $\gamma'(\lambda_1) = \gamma'(\lambda_2) < 1$, while, for $-e < c < 0$, $\gamma'(\xi_3) < 1$.

(If $c = -e$, then $\gamma'(\xi_3) = \gamma'(-1) = 1$. A detailed examination of the behaviour of the points in the neighbourhood of ξ_3 can be carried out, but we avoid the necessity for this as will be seen in the proof of Lemma 4 a.)

The next lemma completes the necessary preliminary results.

LEMMA 4. *When $-e < c < 0$, there exists for the substitution $(x|ce^x)$ one cycle only (ζ_1, ζ_2) of order two for which the two points ζ_1, ζ_2 are distinct and for which both lie within the strip $|\mathrm{im}(x)| < 2\pi$.*

When $c \leqslant -e$, no such cycle exists.

If Lemma 4 is true, it follows from Lemmas 2 and 4 that equation (4.4) has roots ζ_1, ζ_2, ξ_3 when $-e < c < 0$, and $\lambda_1, \lambda_2, \xi_3$ when $c \leqslant -e$. (These are not, of course, the only roots of the equation which lie in $|\mathrm{im}(x)| < 2\pi$. It is easily shown that an infinite number of roots lie within the strip, but all other than those mentioned above correspond to cycles of order two of $(x|ce^x)$ of which one point only lies within the strip.)

Suppose that ζ_1, ζ_2 satisfy

$$0 < \mathrm{im}(\zeta_1) < 2\pi, \qquad -2\pi < \mathrm{im}(\zeta_2) < 0. \tag{4.5}$$

We consider the transformation

$$x = \phi(X) \equiv \log_0\{c^{-1}\log_{-1}(X/c)\},$$

where c is real and, by the definition of the notation previously given, if $x_1 = \log_{-1}(X/c)$, then $-2\pi < \mathrm{im}(x_1) \leqslant 0$ and $0 \leqslant \mathrm{im}(x) < 2\pi$.

If there exist unique double points β_1, β_2 of the transformation such that $0 < \mathrm{im}(\beta_1) < 2\pi$ and $-2\pi < \mathrm{im}(\beta_2) < 0$, then $\beta_3 = \log_{-1}(\beta_1/c)$ satisfies $-2\pi < \mathrm{im}(\beta_3) < 0$ and $\log_0(\beta_3/c) = \beta_1$, so that β_3 is also a double point of the transformation and must be identical with β_2. Hence (β_1, β_2) form a unique cycle of order two of $(x|ce^x)$ and are therefore identical with (ζ_1, ζ_2). An alternative statement of the lemma, when ζ_1, ζ_2 satisfy (4.5), would therefore be

LEMMA 4 a. *The equation $X = \phi(X)$ has no complex roots when $c \leqslant -e$ and two complex roots only*

$$\zeta_1 \text{ and } \zeta_2 \quad (0 < \mathrm{im}(\zeta_1) < 2\pi \text{ and } -2\pi < \mathrm{im}(\zeta_2) < 0)$$

when $-e < c < 0$.

To prove this we first suppose that $0 < \mathrm{im}(X) < 2\pi$ and $c < -e$. A modified rectangular contour $ABCD$ is defined as follows.

AB lies along the real axis but excludes by small semicircles the negative real roots $\lambda_1, \xi_3, \lambda_2$, of $X = \phi(X)$ and the singularities of $\phi(X)$ at $X = 0$ and $X = c$; CD lies along the line $\text{im}(X) = 2\pi$; DA lies along $\text{re}(X) = -S < ce^{-c}$; BC lies along the line $\text{re}(X) = K > K_0$, where K_0 is such that, for all $K > K_0$, $0 < \text{re}(x) < K$ for any X on BC.

To show that BC can be drawn to satisfy this condition, let

$$X = K+iv = re^{i\alpha} \quad (0 \leqslant v \leqslant 2\pi;\ 0 \leqslant \alpha < \tfrac{1}{2}\pi),$$

so that

$$x_1 = \log_{-1}(X/c) = \log(-r/c)+i(\alpha-\pi) = r_1 e^{i\alpha_1} \text{ (say)} \quad (-\pi < \alpha_1 < 0)$$

and
$$x = \log_0 c^{-1} r_1 e^{i\alpha_1} = \log(-r_1/c)+i(\alpha_1+\pi).$$

We see that the condition $0 < \text{re}(x) < K$ is equivalent to

$$0 < \log(-r_1/c) < K,$$

that is $-c < r_1 < -ce^K$ and, since $r_1 = |\log(-r/c)+i(\alpha-\pi)|$ and $r = |K+iv|$, this is satisfied for all $K >$ some K_0. Also

$$ce^{-c} < c < \lambda_1 < \xi_3 < \lambda_2 < 0,$$

so that the two singularities of $\phi(X)$ and the three roots of $X = \phi(X)$ necessarily all lie between A and B.

We now consider the change in the argument of $x-X$ as X passes round contour $ABCD$. Along BC the vector is always directed towards the interior of the contour. Since for X on CD, x necessarily satisfies $0 < \text{im}(x) < 2\pi$ (although it does not necessarily lie within $ABCD$), the vector is directed towards the interior of the contour all along CD. Similarly by the definition of AD, if X lies on AD, $\text{re}(x) > 0$ $0 < \text{im}(x) < 2\pi$, so that the vector is again directed inwards.

Under the transformation $x = \phi(X)$ points $X < c$ on AB become points satisfying $\text{im}(x) = \pi$; if $c < X < \lambda_1$, then $x < X$; each of the segments $\lambda_1 < X < \xi_3$ and $\xi_3 < X < \lambda_2$ corresponds to itself, individual points moving towards ξ_3; if $\lambda_2 < X < 0$, then $x > X$; if $X > 0$, x is again complex. Points near $X = c$ are moved far to the left along the strip and those near $X = 0$ far to the right. By Lemma 3, λ_1 and λ_2 are attractive double points of $(x|(c\exp)^2x)$ and hence repulsive double points of $(X|\phi(X))$ so that on the semicircles surrounding these the vector is directed away from the points, while on the semicircle surrounding the attractive double point ξ_3 the vector is directed towards the point.

Combining these results we see that the total change in $\arg(x-X)$ as X passes round the contour is zero, so that no roots of $\phi(X) = X$ are enclosed. Further, by definition of the contour, CB can be moved as far

to the right and AD as far to the left, as we choose, so that there are no roots in the whole strip $0 < \mathrm{im}(X) < 2\pi$.

Again by Lemmas 2 and 3, if $-e < c < 0$, ξ_3 is the only real root and is a repulsive double point of $(X|\phi(X))$. The contour $ABCD$ is defined as before except that the only semicircles required are those to exclude $X = c$, ξ_3, and 0. There is now a change of 2π in $\arg(x-X)$ as X passes round the contour, so that there exists only one complex root $X = \zeta_1$ of $X = \phi(X)$ which satisfies $0 < \mathrm{im}(X) < 2\pi$.

If $c = -e$, then $\xi_3 = -1$ and examination of the behaviour of points on the semicircle surrounding -1 is lengthy. We observe, however, that a small circle surrounding -1 encloses a third-order zero of $x-(c\exp)^2x$, so that the change in argument of $x-(c\exp)^2x$ as x passes round the circle would be 6π, and hence, by symmetry about the real axis, 3π in passing round a semicircle based on the axis. Thus, in our previous notation, the change in $\arg(x-X)$ as X passes round the semicircle would be -3π and the proof that there are no complex roots of $X = \phi(X)$ in the strip $0 < \mathrm{im}(X) < 2\pi$ when $c = -e$ is readily completed.

Similar arguments to the above, both for $c \leqslant -e$ and for $-e < c < 0$ but with $-2\pi < \mathrm{im}(X) < 0$ and a contour defined in the strip below the axis, prove the existence of the unique root ζ_2 $(-2\pi < \mathrm{im}(\zeta_2) < 0)$. This completes the proof of Lemma 4a.

Similarly it can be shown that there is no cycle of order two of $(x|ce^x)$ for which the two distinct points both lie in $0 < \mathrm{im}(x) < 2\pi$ or both lie in $-2\pi < \mathrm{im}(x) < 0$, for

$$x = \log_0\{c^{-1}\log_0(X/c)\} \quad \text{and} \quad x = \log_{-1}\{c^{-1}\log_{-1}(X/c)\}$$

each have one double point only, which must therefore be the appropriate root of $x = ce^x$. This completes the proof of Lemma 4.

Using the preceding results, I now establish a series of lemmas concerning the roots of the equation $x = (c\exp)^n x$ and the cycles of $(x|ce^x)$.

LEMMA 5. *Corresponding to each set of positive or negative integers or zeros $(h_1, h_2,..., h_n)$ $(h_n \neq 0$ or $-1)$ there exists a unique root*

$$x_n \quad (2h_n\pi < \mathrm{im}(x_n) < 2(h_n+1)\pi)$$

of the equation

$$x = (c\exp)^n x \tag{4.6}$$

and a unique set of points $(x_1, x_2,..., x_n)$ $(2h_r\pi < \mathrm{im}(x_r) < 2(h_r+1)\pi)$ forming a cycle of the n-th order of the substitution $(x|ce^x)$.

We consider an inverse of the substitution $X = (c\exp)^n x$, namely

$$x = \theta(X) \equiv \log_{h_n}[c^{-1}\log_{h_{n-1}}\{c^{-1}\log_{h_{n-2}}...c^{-1}\log_{h_1}(X/c)\}], \tag{4.7}$$

and define a rectangular contour $ABCD$ to have sides AB, CD along the lower and upper boundaries of the strip h_n and DA, BC sufficiently far to the left and right respectively of the imaginary axis for $\theta(X)$ to lie within $ABCD$ for any X on AD or BC.

It is readily confirmed that AD, BC can be chosen to satisfy this requirement, and that there are no zeros or singularities of $\theta(X)-X$ on the boundary of the contour and no singularities enclosed. At every point round the contour, the vector $x-X$ is directed inwards, so that there is a change of 2π in $\arg(x-X)$ as X passes round the contour and one root only of $X = \theta(X)$ is therefore enclosed. Since BC, AD can be moved as far to the right and left respectively as we choose, there is no other root in the strip h_n.

If we call this root x_n, a unique point x_1 in the strip h_1 is defined by

$$x_1 = \log_{h_1}(x_n/c),$$

a unique point x_2 by

$$x_2 = \log_{h_2}(x_1/c),$$

and so on. Finally $x_n = \log_{h_n}(x_{n-1}/c)$. Thus to a given set $(h_1, h_2,..., h_n)$ there corresponds a unique cycle $(x_1, x_2,..., x_n)$, change in any h_r $(1 \leqslant r < n)$ giving rise to a different root of equation (4.6) in the same strip as x_n and to a different cycle of order n of $(x|ce^x)$.

LEMMA 6. *The results of Lemma* 5 *apply if* $h_n = 0$ *or* -1 *provided that* $h_s \neq 0$ *or* -1 *for some* $s < n$.

We apply the argument of Lemma 5 to a similarly defined contour in the strip h_s to prove the existence of a root x_s of equation (4.6) and a cycle $(x_{s+1}, x_{s+2},..., x_n, x_1,..., x_s)$ of $(x|ce^x)$ corresponding to the set $(h_{s+1}, h_{s+2},..., h_n, h_1,..., h_s)$. But every point of the cycle is a root of equation (4.6) corresponding to the appropriate h-set, so that the existence of x_n corresponding to $(h_1, h_2,..., h_n)$ is proved.

LEMMA 7. *The results of Lemma* 5 *apply if* $c > 0$ *and every* h_r *is either* 0 *or* -1 *provided that the* h_r *are not all equal.*

Since the h_r are not all equal, we have for some s $(1 \leqslant s < n)$ either $h_s = -1$, $h_{s+1} = 0$ or $h_s = 0$, $h_{s+1} = -1$. In the former case, if a cycle exists with a point x_s in h_s, then $x_{s+1} = \log_{h_{s+1}}(x_s/c)$ and $\pi < \text{im}(x_{s+1}) < 2\pi$. We can apply the argument of Lemma 5 to a suitably defined rectangular contour in the upper half of the strip h_{s+1} to establish the existence of the root x_{s+1} corresponding to the appropriate h-set, and hence, as in Lemma 6, to establish the existence of x_n.

If $h_s = 0$, $h_{s+1} = -1$, the procedure is similar, by using a rectangular contour two sides of which lie along $\text{im}(X) = -\pi$ and $\text{im}(X) = -2\pi$.

Lemma 8. *When $c > 0$, the only roots of equation* (4.6) *corresponding to the sets* (i) $h_r = 0$ *for all* $r \leqslant n$ *and* (ii) $h_r = -1$ *for all* $r \leqslant n$ *are those roots of the equation* $x = ce^x$ *which satisfy* $|\mathrm{im}(x)| < 2\pi$.

When $h_r = 0$ for all $r \leqslant n$, this lemma is equivalent to the statement that there are no cycles of order n of $(x|ce^x)$ of which all points lie in the strip $0 < \mathrm{im}(x) < 2\pi$, other than that for which (when $c > e^{-1}$)

$$x_1 = x_2 = \ldots = x_n,$$

each being equal to the root of $x = ce^x$.

The proof of the lemma is similar to that of Lemma 4. If $0 < c < e^{-1}$, then $x = \log(x/c)$ has, by Lemma 1, two real roots,

$$\xi_1 \text{ and } \xi_2 \quad (0 < \xi_1 < 1 < \xi_2).$$

If $\eta(x) = \log(x/c)$, then $\eta'(x) = x^{-1}$, so that $\eta'(\xi_1) > 1$ and $\eta'(\xi_2) < 1$, making ξ_1 a repulsive and ξ_2 an attractive double point of $\{x|\log(x/c)\}$. Equation (4.7) now becomes $x = \theta(x) \equiv \log[c^{-1}\log\{c^{-1}\ldots c^{-1}\log(X/c)\}]$, where every logarithm has its imaginary part between 0 and 2π, and we examine the change in $\arg(x-X)$ as X passes round a contour defined, with obvious modifications, as in the proof of Lemma 5. In this case the contour will exclude ξ_1 and ξ_2 when they exist, the origin, and the $n-1$ points c, ce^c, c^{ce^c},.... We find no change in $\arg(x-X)$ as X passes round the contour when $0 < c < e^{-1}$ but a change of 2π when $c > e^{-1}$. Hence there is no root of equation (4.6) enclosed when $0 < c < e^{-1}$ and one root only (which must therefore be that of $x = ce^x$) when $c > e^{-1}$.

A similar argument applies when $h_r = -1$ for all $r \leqslant n$ using a contour in the strip below the real axis.

Lemma 9. *The results of Lemma* 5 *apply if* $c < 0$ *and every* $h_r = 0$ *or* -1, *including the case when every* $h_r = 0$ *or every* $h_r = -1$ *but provided, if* n *is even, that they do not alternately take the values* 0, -1.

If the h_r do not alternately take the values 0, -1, there exists in every case mentioned h_s such that $h_s = 0$, $h_{s+1} = 0$, or $h_s = -1$, $h_{s+1} = -1$. If the h_r alternate but n is odd, either $h_1 = 0$ and $h_n = 0$ or $h_1 = -1$ and $h_n = -1$. If $h_s = 0$, if X_s is any point in the corresponding strip, and if $X_{s+1} = \log_{h_{s+1}}(X_s/c)$ (we replace $s+1$ by 1 if $s = n$ and $h_1 = 0$), then $\pi < \mathrm{im}(X_{s+1}) < 2\pi$ since $c < 0$ and an argument similar to that of Lemma 7 establishes the existence of the root x_{s+1}. If $h_s = -1$, $h_{s+1} = -1$, or $h_n = -1$, $h_1 = -1$, the proof is again similar, by using a contour in $-2\pi < \mathrm{im}(x) < -\pi$.

LEMMA 10. *If $-e < c < 0$, the negative root ξ_3 of $x = ce^x$ is the only real root of equation* (4.6). *If also n is even, the only complex roots of the equation corresponding to the sets*

$$(h_1, h_2, h_3, \dots, h_n) \equiv (-1, 0, -1, \dots, 0) \text{ and } (0, -1, 0, \dots, -1) \quad (4.8)$$

are the roots ζ_1 and ζ_2 of equation (4.4).

If $c \leqslant -e$ and n is even, the roots λ_1, ξ_3, and λ_2 of (4.4) *are the only real roots of* (4.6), *and there are no complex roots corresponding to the sets* (4.8).

It is obvious that the roots mentioned are all roots of (4.6) and that the equation has no other real roots. The problem is to show that there are no other complex roots corresponding to the sets (4.8). The argument of Lemma 4 extends at once, the contour being similarly defined, but excluding λ_1 and λ_2 (when $c < -e$), ξ_3, the origin, and the $n-1$ points $c, ce^c, ce^{ce^c}, \dots$. Using the results of Lemma 3, we find that, as X passes round the contour, there is no change in $\arg(x-X)$, where

$$x = \phi(X) \equiv \log_0[c^{-1}\log_{-1}\{c^{-1}\log_0 \dots \log_{-1}(X/c)\}]$$

when $c < -e$, but a change of 2π when $-e < c < 0$: that is, when λ_1, λ_2 do not exist and ξ_3 is repulsive. Thus, when $-e < c < 0$, there is one root only, which must be ζ_1, within the contour. Similarly, using a contour in the strip below the real axis, we establish the existence of ζ_2 when $-e < c < 0$ as the only root corresponding to $(0, -1, 0, \dots, -1)$. Lemmas 5–10 together establish Theorem 1.

5. Additional results

So far we have considered real values of c only. If c is complex, the absence of real roots of equation (4.6) is immediately obvious and the problem is considerably simplified. I establish the following result.

THEOREM 2. *Corresponding to each set of positive or negative integers or zeros $(h_1, h_2, \dots, h_n)$ there exists a unique root x_n of the equation*

$$x = (c\exp)^n x \quad (c = \rho e^{i\sigma};\ \sigma \neq 0),$$

and a unique set of points $(x_1, x_2, \dots, x_n)$ $(2h_r\pi - \sigma < x_r < 2(h_r+1)\pi - \sigma)$ forming a cycle of the n-th order of the substitution $(x|ce^x)$.

A rectangular contour $ABCD$ is drawn in the strip

$$2h_n\pi - \sigma < \operatorname{im}(x) < 2(h_n+1)\pi - \sigma$$

with sides AB, CD along the lower and upper boundaries of the strip and with BC, AD defined as in the proof of Lemma 5. The contour is now modified by excluding, by small circles joined by cuts to the boundary of the rectangle, any of the points $0, c, ce^c, ce^{ce^c} \dots$ which may be enclosed.

The remainder of the proof is an obvious application of the method of proof of Lemma 5.

The distribution of the roots of $x = ce^x$ when c is complex follows at once as a particular case of the above theorem. Further information in this particular case can, however, be obtained by the method by which Lemma 1 of this paper was established in (**4**). The results are given in the following theorem.

THEOREM 3. *The roots of the equation* $x = ce^x$, *where* $x = u+iv$, $c = \rho e^{i\sigma}$ $(0 < \sigma < \pi)$, *lie one in each strip*

$$2p\pi-\sigma < v < (2p+1)\pi-\sigma \quad (p = 1,2,3,\ldots), \tag{5.1}$$

one in each strip

$$(2p+1)\pi-\sigma < v < 2(p+1)\pi-\sigma \quad (p = -1,-2,-3,\ldots), \tag{5.2}$$

and one in the strip $\qquad 0 < v < \pi-\sigma,$

each root being at the intersection of the curve (i) $v = \pm(\rho^2e^{2u}-u^2)^{\frac{1}{2}}$ *and the appropriate branch of* (ii) $u = v\cot(v+\sigma)$.

The proof is omitted since it follows closely that for real c given in (**4**).

Curves (i) and (ii) closely resemble the diagram in (**4**), (i) differing from $v = \pm(c^2e^{2u}-u^2)^{\frac{1}{2}}$ only in the replacement of c by $|c|$, and the branches of (ii) lying one in each strip of width π as for $u = v\cot v$, the strips now being displaced downward by an amount σ. For real c, intersections of $v = \pm(c^2e^{2u}-u^2)^{\frac{1}{2}}$ with successive branches of $u = v\cot v$ were alternately roots for $c > 0$ and $c < 0$. Similarly now, the intersections of (i) with the branches of (ii) lying in the strips between strips (5.1) and between strips (5.2) are the location of roots $x = ce^x$ when $\arg c = \pi+\sigma$ but not when $\arg c = \sigma$.

I am indebted to Professor E. M. Wright for suggesting this problem and for his advice in the preparation of the paper.

REFERENCES

1. G. Eisenstein, 'Entwicklung von $\alpha^{\alpha^{\alpha^{\cdot^{\cdot}}}}$', *J. für Math.* 28 (1844), 49–52.
2. L. Euler, 'De formulis exponentialibus replicatis', *Opera Omnia* (i), 15 (Leipzig and Berne, 1927).
3. D. Gravé, 'Sur les expressions dites surpuissances', *Nouv. Ann. de Math.* (3) 17 (1898), 80–91.
4. N. D. Hayes, 'Roots of the transcendental equation associated with a certain difference-differential equation', *J. of London Math. Soc.* 25 (1950), 226–32.
5. E. M. Lémeray, 'Le quatrième algorithme naturel', *Proc. Edinburgh Math. Soc.* 16 (1898), 13–35.
6. E. M. Wright, 'Iteration of the exponential function', *Quart. J. of Math.* (Oxford), 18 (1947), 228–35.

A METHOD OF FINDING THE CRITICAL LATTICES OF SPHERES CONTAINING THE ORIGIN

By H. AP SIMON (*Oxford*)

[Received 21 March 1951; revised 25 August 1951]

1. Introduction

In this note I give details of a method which I have applied† to the problem of finding the critical lattices of the 'Off-Centre Sphere', or region bounded by a sphere whose centre does not coincide with the lattice origin. Specifically, the region considered is the star-body R, the union of the regions bounded by the spheres

$$(S_1)\quad (x-k)^2+y^2+z^2 = 1,$$
$$(S_2)\quad (x+k)^2+y^2+z^2 = 1,$$

where $0 < k < 1$. (If $k \geqslant 1$ the origin is not an interior point of R, and we can find admissible lattices of arbitrarily small determinant.) The final result is given as

Theorem. *The star-body R has an infinity of critical lattices, of determinant*

$$\Delta(R) = \tfrac{1}{2}(1-k^2)\{\sqrt{3}k+\sqrt{(2+k^2)}\},$$

and defined by

$$\left.\begin{aligned} x &= \sqrt{\tfrac{1}{3}}\{\sqrt{3}k+\sqrt{(2+k^2)}\}\xi \\ y &= \sqrt{(1-k^2)}\{\sqrt{\tfrac{1}{3}}\xi\sin\theta+\eta\cos(\theta+\tfrac{2}{3}\pi)+\zeta\cos\theta\} \\ z &= \sqrt{(1-k^2)}\{-\sqrt{\tfrac{1}{3}}\xi\cos\theta+\eta\sin(\theta+\tfrac{2}{3}\pi)+\zeta\sin\theta\} \end{aligned}\right\}. \tag{1}$$

2.1. Certain definitions are first necessary. I define a *grid* G as the set of points $aA+bB+cC$, where a, b, c take the values $-1, 0, 1$ independently and the points A, B, C are independent. The points $\pm A$, $\pm B$, $\pm C$ I call the *face-centres* of the grid, and the lattice having A, B, C as a basis I call the *equivalent lattice* of the grid (it will be noticed that the points of the grid are points of its equivalent lattice); I define the *determinant* $d(G)$ of G as the determinant of its equivalent lattice. I call a grid *admissible* with respect to a body K if (i) no grid point other than O is an inner point of K, and (ii) the face-centres of the grid lie on the boundary of K. I denote the minimum of $d(G)$ over all admissible grids by $\nabla(K)$, and call any admissible grid of determinant $\nabla(K)$ *critical*.

† As part of a thesis accepted for the degree of D.Phil. at Oxford.

Quart. J. Math. Oxford (2), 3 (1952), 91–93

2.2. Let K be a closed finite star-body formed by the union of the bodies K_1 and K_2, where K_1 is a strictly convex body containing the origin as an inner point and K_2 is the reflection of K_1 in the origin. Let Λ be a critical lattice of K; then three independent points of Λ lie on the boundary of K.

LEMMA 1. *Any two points of Λ on the boundary of K form part of a basis of Λ.*

Since K is composed of the union of K_1 and its reflection in the origin, three independent points of Λ lie on the boundary of K_1. Call A and B any two of these points, and call L_1 the two-dimensional lattice composed of the lattice-points of Λ in the plane AOB, and D the domain in which the plane AOB intersects K_1.

Since D is strictly convex having O as an inner point, and A and B lie on the boundary of D, no lattice-point of L can be an inner point of the triangle OAB or an inner point of a side of this triangle. Hence† the determinant of the lattice having basis A, B is $d(L_1)$, and so‡ A and B form a basis of L_1.

But, if A and B form a basis of L_1, so do A and $-B$, and so any two points A', B' of Λ on the boundary of K form a basis of the two-dimensional lattice consisting of the points of Λ in the plane $A'OB'$. To complete the proof of the lemma it is necessary to show that A', B' must necessarily form part of a basis of Λ. This will follow from the following, the proof of which is trivial.

LEMMA 2. *Let Λ be a three-dimensional lattice and P, Q any two points of Λ. Call L the two-dimensional lattice consisting of the points of Λ in the plane POQ. Then, if P, Q form a basis of L, they form part of a basis of Λ.*

2.3. LEMMA 3. *If K is a body of the type defined in* § 2.2, *then*

$$\nabla(K) \leqslant \Delta(K).$$

By Lemma 1 there are two points of Λ (any critical lattice of K) on the boundary of K which form part of a basis of Λ; call them A and B, and call L_0 the two-dimensional lattice generated by A and B.

The lattice Λ consists of the points of lattices $L_{\pm n}$ parallel and congruent to L_0 and distant cn from O, where c is a positive constant and n takes all non-negative integer values.

† Hardy and Wright, *The Theory of Numbers* (Oxford 1945), Theorem 34 (ii).

‡ Ibid. Theorem 33.

(i) If a point C of $L_{\pm 1}$ lies on the boundary of K, then Λ is the equivalent lattice of the admissible grid G having face-centres A, B, C, and so $\nabla(K)$ exists and
$$\nabla(K) \leqslant d(G) = d(\Lambda) = \Delta(K).$$

(ii) If no point of $L_{\pm 1}$ lies on the boundary of K, then we may reduce the value of c, and hence also $d(\Lambda)$, keeping A and B fixed until a point C' of $L_{\pm 1}$ lies on the boundary of K and no point of $L_{\pm 1}$ lies inside K. Call Λ' the lattice generated by A, B, C' (Λ' is not admissible) and G' the grid having face-centres A, B, C'. Since no point of L_0, $L_{\pm 1}$ other than O lies inside K, and A, B, C' lie on the boundary of K, it follows that G' is admissible. Hence $\nabla(K)$ exists and
$$\nabla(K) \leqslant d(G') = d(\Lambda') < d(\Lambda) = \Delta(K).$$
We have then in both cases that $\nabla(K)$ exists and $\nabla(K) \leqslant \Delta(K)$. This proves the lemma.

2.4. Let Λ' be the equivalent lattice of a critical grid G. Then, if Λ' is admissible,
$$\nabla(K) = d(G) = d(\Lambda') \geqslant \Delta(K).$$
But, by Lemma 3, $$\nabla(K) \leqslant \Delta(K).$$
Hence, if Λ' is admissible,
$$d(\Lambda') = \nabla(K) = \Delta(K),$$
and so Λ' is critical. We have then the eventual result:

LEMMA 4. *If K is a star-body of the type defined in* § 2.2, *then the equivalent lattice of a critical grid is critical if it is admissible.*

3. The problem of finding the critical lattices of R is thus changed to the problem of finding its critical grids, and then testing them to see whether their equivalent lattices are admissible. The solution of this problem involves much tedious enumeration of cases, but it is eventually found that any critical grid of R has face-centres A, B, C such that A, B and $A+B$ lie on the circle of intersection of S_1 and S_2, and C, $C+A$, $C+A+B$ lie on the boundary of S_1. It is then easily verified that the equivalent lattices of such grids are defined by (1), and that these lattices are admissible. It follows by Lemma 4 that they are therefore critical, and so the theorem is proved.

IDEMPOTENT OPERATORS ON A VECTOR SPACE

By H. S. ALLEN (*London*)

[Received 8 March 1951]

1. The operators considered belong to an algebra of linear operators on a vector space into itself. Suppose A and B operators, I the identity and $AB = I$. The following results are established. *The null space of A is the range of the operator $I-BA$. The set of left inverses of B is the set of all left multiples of elements of a group of automorphisms by the operator A. The set of idempotents with the same range as B is the set of all left multiples of the set of left inverses of B by the operator B.*

Finally I apply these results to the theory of infinite matrices and to the homogeneous system of linear equations in infinitely many unknowns.

2. Let $\mathfrak{X}$ be an algebra of linear operators on a vector space α into itself and let $\mathfrak{X}$ contain the identity. The set of all operators in $\mathfrak{X}$ which have an inverse in $\mathfrak{X}$ is a multiplicative group $\mathfrak{G}$. The set of all operators in $\mathfrak{G}$ which leave all points of a subspace β invariant is a subgroup $g(\beta)$ of $\mathfrak{G}$. I shall denote the set of all idempotent operators in $\mathfrak{X}$ with range β by $P(\beta)$. If $A \in P(\beta)$ and $I-A \in P(\mu)$, then $Ax = x$ if and only if $x \in \beta$, and $Ax = 0$ if and only if $x \in \mu$. The intersection $\beta \cap \mu = 0$, and every $x \in \alpha$ has a unique decomposition $x = u+v$ where $u \in \beta$ and $v \in \mu$. Subspaces of α which satisfy these two conditions are called *complementary* subspaces† of α. An operator A is said to *project* α on β if and only if $A \in P(\beta)$.

3.1. *Two linear operators A and B are idempotent with the same range if and only if $AB = B$ and $BA = A$.*

If A and B are idempotents with the same range, then $(AB)x = Bx$ for all x since Bx is in the range of A. Conversely let $AB = B$ and $BA = A$. Then $(AB)A = BA = A$ and $A(BA) = A^2$. Thus $A^2 = A$ and $B^2 = B$. If β_1 and β_2 are the ranges of A and B and $x \in \beta_1$, we have $x = Ax$ and $Bx = (BA)x = Ax = x$ and therefore $x \in \beta_2$. Thus $\beta_1 = \beta_2$.

† F. J. Murray, 'On complementary manifolds and projections in spaces L_p and l_p', *Trans. American Math. Soc.* 41 (1937) 138. See also G. Birkhoff and S. MacLane, *A Survey of Modern Algebra* (New York, 1949) 180, 303.

Quart. J. Math. Oxford (2), 3 (1952), 94–97

3.2. *If $A \in P(\beta)$ and $B \in \mathfrak{G}$, then BAB^{-1} projects α on λ if and only if $B\beta = \lambda$.*

We have $(BAB^{-1})^2 = BA^2B^{-1} = BAB^{-1}$ and so BAB^{-1} is an idempotent. Let λ be the range of BAB^{-1}. If $x \in \lambda$, we have $(BAB^{-1})x = x$ and $A(B^{-1}x) = B^{-1}x$. Hence $B^{-1}x \in \beta$ and $B^{-1}\lambda \subseteq \beta$.

If $y \in \beta$ and $By = u$, then $Ay = y$, i.e. $A(B^{-1}u) = y$ and

$$(BAB^{-1})u = By = u.$$

It follows that $u \in \lambda$ and $B\beta \subseteq \lambda$. Hence $B\beta = \lambda$.

To prove the converse, suppose that $B\beta = \lambda$. If $y \in \alpha$, then $A(B^{-1}y) \in \beta$, and hence $(BAB^{-1})y \in \lambda$. If $x \in \lambda$, then $A(B^{-1}x) = B^{-1}x$ and $(BAB^{-1})x = x$. Thus $(BAB^{-1})\alpha = \lambda$.

3.3. *If β and μ are complementary subspaces of α and $A \in \mathfrak{G}$, then the subspaces $A\beta$ and $A\mu$ are complementary.*

Let $A\beta = \lambda$ and $A\mu = \gamma$. If $x \in \lambda \cap \gamma$, we have $x = Af$ where $f \in \beta$ and $x = Ah$ where $h \in \mu$. Then $A(f-h) = 0$, and hence $f = h$. It follows that $f \in \beta \cap \mu$, and hence that $f = 0$ and $x = 0$. If y is an arbitrary point of α, we have $A^{-1}y = u+v$, where $u \in \beta$ and $v \in \mu$. Since $y = Au+Av$, the result follows.

3.4. *If* (i) *A and B belong to $P(\beta)$,* (ii) *$I-A \in P(\mu)$,* (iii) *$I-B \in P(\theta)$, then the operator $X = I+A-B$ has the properties (a) $X \in g(\beta)$, (b) $X\mu = \theta$.*

Each of the spaces μ and θ is a complementary space to β. We have $AB = B$ and $BA = A$, by § 3.1. It follows by direct multiplication that

$$(I+A-B)(I-A+B) = I = (I-A+B)(I+A-B).$$

Hence $X^{-1} = I-A+B$ belongs to $\mathfrak{X}$, and therefore $X \in \mathfrak{G}$. Since $Ax = x = Bx$ when $x \in \beta$, (a) follows. If $x \in \mu$, then $Ax = 0$ and $Xx = (I-B)x \in \theta$. Thus $X\mu \subseteq \theta$. If $y \in \theta$, then $y = Xf$, where $f \in \alpha$, by (a), and $f = u+v$ where $u \in \beta$ and $v \in \mu$. We have $Xu = u$ and $Av = 0$ and hence $y = Xf = u+(I-B)v$. The point $(I-B)v \in \theta$, and hence $u = y-(I-B)v \in \theta$. Thus $u \in \beta \cap \theta$ and it follows that $u = 0$. Hence $f = v \in \mu$. This proves (b).

When S is a subset of $\mathfrak{X}$, the set of all operators AX, where $X \in S$, will be denoted by AS.

3.5. *If $A \in P(\beta)$, then $P(\beta) = Ag(\beta)$.*

Suppose that $B \in P(\beta)$ and $X = I-A+B$; then $X \in g(\beta)$ by § 3.4. We have $AX = A(I-A)+AB = AB = B$ by § 3.1. It follows that $B \in Ag(\beta)$.

If $C \in g(\beta)$, then $C(Ax) = Ax$ for every x, and hence $CA = A$. Therefore $(AC)^2 = A(CA)C = A^2C = AC$. We have $C\alpha = \alpha$ because $C \in \mathfrak{G}$, and hence $(AC)\alpha = A\alpha = \beta$. Thus $AC \in P(\beta)$. This proves the result.

3.6. *If $A \in P(\beta)$ and $B \in P(\gamma)$, then there are operators $C \in P(\beta)$, $I-C \in P(\gamma)$ if and only if there are operators $X \in g(\beta)$, $Y \in g(\gamma)$ such that $AX+BY = I$.*

If C exists, we have $AC = C$ and $B(I-C) = I-C$, by § 3.1. Let $X = I-A+C$ and $Y = I-B+(I-C) = 2I-B-C$. It follows from § 3.4 that $X \in g(\beta)$ and $Y \in g(\gamma)$, and we have

$$AX+BY = C+B(I-C) = I.$$

Conversely, if $AX+BY = I$, where $X \in g(\beta)$, $Y \in g(\gamma)$, then by § 3.5, $C = AX \in P(\beta)$ and $I-C = BY \in P(\gamma)$.

3.7. *If B is a right inverse of A and $(I-BA)\alpha = \theta$, then θ is the space of all $x \in \alpha$ such that $Ax = 0$.*

We have $A(Bx) = x$ for every x, and hence $A\alpha = \alpha$. Let $B\alpha = \beta$. Since $(BA)^2 = B(AB)A = BA$ and $(BA)\alpha = B\alpha = \beta$, it follows that $BA \in P(\beta)$. If $x \in \theta$, we have $(BA)x = 0$ and $A(BA)x = Ax = 0$. Conversely, if $Ax = 0$, then $(BA)x = 0$ and $x \in \theta$.

The set of all left inverses in $\mathfrak{X}$ of an operator $A \in \mathfrak{X}$ will be denoted by $l(A)$.

3.8. *If* (i) $A\alpha = \beta$, (ii) $B \in l(A)$, *then*

$$(a)\ l(A) = Bg(\beta), \qquad (b)\ P(\beta) = Al(A).$$

If $C \in g(\beta)$, then $C(Ax) = Ax$ for every x and so $CA = A$. Hence $(BC)A = B(CA) = BA = I$. Thus $BC \in l(A)$. Suppose $X \in l(A)$. When we replace A by X and B by A in § 3.7, we obtain $AX \in P(\beta)$ and in particular $AB \in P(\beta)$. By § 3.4 the operator $Y = I+AB-AX \in g(\beta)$. We have $XY = X+(XA)B-(XA)X = B$ and therefore $X = BY^{-1}$. This proves (a). We have proved that $Al(A)$ is a subset of $P(\beta)$. If $E \in P(\beta)$, then $E(Ax) = Ax$ for every x and so $EA = A$. Hence

$$I = BA = B(EA) = (BE)A,$$

and therefore $BE \in l(A)$. It follows that $A(BE) \in P(\beta)$. We have $(AB)E = E$ by § 3.1 and hence $E = A(BE) \in Al(A)$. This proves (b).

3.9. *If $B \in l(A)$, then $kI+AB \in \mathfrak{G}$ for every scalar k except* 0 *and* -1 (*the former not excluded if $AB = I$*).

Let $A\alpha = \beta$, then $AB \in P(\beta)$ by § 3.8. If $AB = E$, we have $x = u+v$

where $u = Ex$, $v = (I-E)x$ and $(kI+E)x = (1+k)u+kv$. Hence the inverse of $kI+E$ is given by

$$(kI+E)^{-1} = (1+k)^{-1}E+k^{-1}(I-E)$$

and belongs to $\mathfrak{X}$ by the closure properties of the algebra.

4. Applications to theory of infinite matrices

Let α be a perfect sequence space† and $\Sigma(\alpha)$ the algebra of all infinite matrices which map α into itself.‡ The set of all matrices in $\Sigma(\alpha)$ which have a two sided reciprocal (inverse) in $\Sigma(\alpha)$ is a multiplicative group $G(\alpha)$. Suppose A and B belong to $\Sigma(\alpha)$ and $AB = I$. It follows from § 3.7 that all solutions in α to the homogeneous system of linear equations $Ax = 0$ are obtained by application of the matrix $I-BA$ to arbitrary points of α. If $B\alpha = \beta$ and $g(\beta)$ is the group of all $X \in G(\alpha)$ such that $Xx = x$ when $x \in \beta$, then, by § 3.8, $Ag(\beta)$ is the set of left-hand reciprocals in $\Sigma(\alpha)$ of B. We see from § 3.9 that, if $BA \neq I$, then $S+BA \in G(\alpha)$ for every scalar matrix S except 0 and $-I$.

† R. G. Cooke, *Infinite Matrices and Sequence Spaces* (London, 1950), 275.

‡ G. Köthe and O. Toeplitz, 'Lineare Räume mit unendlichvielen Koordinaten und Ringe unendlicher Matrizen', *J. für Math.* 171 (1934), 193–226 (204).

ON THE CLASS GROUP OF RELATIVELY ABELIAN FIELDS

By A. FRÖHLICH (*Leicester*)

[Received 29 March 1951]

1. Introduction

IN this paper the class groups of relatively Abelian algebraic number fields will be investigated. Very little is known at present of the structural properties of class groups of fields with given Galois group.

A class group in a relatively self-conjugate number field can be considered as a 'representation module' of its relative Galois group, provided that the unit class remains invariant under all substitutions of this Galois group. The use of this relationship in a study of class groups was first suggested to me by Professor H. Heilbronn.

In this approach no restriction on the fields in terms of the roots of unity is necessary. After this paper, however, I shall restrict myself, however, to fields whose relative degree is a power of a prime l, and in order to avoid exceptional cases I assume $l \neq 2$. Most of the work will be expressed in terms of class groups and class fields; the equivalent formulations in terms of the group of units will, for briefness, in general be omitted.

In this paper I begin with an analysis of the relationship of class groups and Galois groups for relatively self-conjugate fields. I then deal with groups of ideal classes in relatively Abelian fields whose order is prime to the degree of the field.

The theory of class fields forms the basis of the present work [cf. (**1**)]. I recall here that an 'ideal group' $\mathfrak{H}^{(\mathfrak{a})}$ in a field K is supposed to contain only ideals prime to a fixed ideal $\mathfrak{a}$ in K, and to contain all ideals (α) such that the number α in K satisfies $\alpha \equiv 1 \pmod{\mathfrak{a}}$. A class group $\mathfrak{C} = \mathfrak{C}^{(\mathfrak{a})}$ is defined as a quotient group $\mathfrak{A}^{(\mathfrak{a})}/\mathfrak{H}^{(\mathfrak{a})}$, where $\mathfrak{A}^{(\mathfrak{a})}$ is the group of all ideals in K prime to $\mathfrak{a}$. The elements of the group $\mathfrak{C}$ are the ideal classes in K. Ideal groups and class groups are throughout assumed to be defined in this manner. As usual in class field-theory I shall not in general state explicitly the ideal $\mathfrak{a}$; it can always be suitably chosen. This implies in particular that two ideal groups $\mathfrak{H}$ and $\mathfrak{H}'$ can be considered as essentially equal if an ideal $\mathfrak{b}$ exists such that $\mathfrak{H} \wedge \mathfrak{A}^{(\mathfrak{b})} = \mathfrak{H}' \wedge \mathfrak{A}^{(\mathfrak{b})}$.

If $\bar{K} = K(\mathfrak{H})$ is a class field of K for the ideal group $\mathfrak{H}$, $\bar{K}$ and $\mathfrak{H}$ will be said to 'belong to each other'. The conductor of $\bar{K}$ over K will be denoted by $\mathfrak{f}(\bar{K}/K) = \mathfrak{f}(\mathfrak{H})$.

I also assume a knowledge of Galois theory of algebraic number fields [cf. (**2**)] and of the theory of Abelian groups with operators [cf. (**2**)], including the results of (**3**).

I should like here to express my gratitude to Professor H. Heilbronn for many valuable discussions and for his continued interest in the progress of this work.

Notation

Small italics denote rational numbers and integers, in particular p, l prime numbers, and always $l \neq 2$. Algebraic integers and numbers will be denoted by small Greek letters, and ideals in algebraic number fields by small Gothic letters. I use italic capitals for ideal classes, in particular I for the unit class, and Gothic capitals for ideal groups and class groups, in particular $\mathfrak{J}$ for the group of principal ideals and $\mathfrak{A}$ for the group of all ideals—where, whenever necessary, ideals not relatively prime to some fixed ideal are excluded.

Greek capitals are in general used for Galois groups; the letter $\mathfrak{o}$ denotes their group ring, and $\mathfrak{r}$, $\mathfrak{s}$, $\mathfrak{t}$ ideals in $\mathfrak{o}$. Elements of Galois groups and their group rings are denoted by small Greek letters. The operator $\sum_{\delta \in \Delta} \delta$ will be written as $N(\Delta)$ or $N(\delta)$. The group generated by elements $\delta_1, \delta_2,...$ is denoted by $\{\delta_1, \delta_2,...\}$.

Any other notations which are not evident will be explicitly defined.

As is usual in class-field theory, groups of ideal classes are written multiplicatively, and therefore, unlike (**3**), operators are, in general, written exponentially, which also implies that the order of the operators is reversed: $C^{\delta\delta'}$ stands for the transform of C first by δ and then by δ'.

2. Galois group and class group

Let K be an algebraic number-field, self-conjugate over a subfield k with relative Galois group Δ, and let $\mathfrak{H}$ be an ideal group invariant under all substitutions of Δ. We consider the class group $\mathfrak{C} = \mathfrak{A}/\mathfrak{H}$. A set $\mathfrak{G}$ of ideal classes in $\mathfrak{C}$ will be called *invariant* if $\mathfrak{G}$ contains with each class all its conjugates under Δ; and $\mathfrak{G}$ will be called *fixed* under a subgroup Ω of Δ if every substitution of Ω leaves every class in $\mathfrak{G}$ invariant. An invariant group $\mathfrak{G}$ of ideal classes is said to be an *elementary* group† if $\mathfrak{G}$ is not the direct product of two invariant proper subgroups. Restating some elementary theorems on Abelian groups and some of the results of (**3**) in terms of the class group $\mathfrak{C}$ we have:

† It should be noted that the term 'elementary group' has a meaning in this paper which is different from its usual one in abstract group theory.

A 1. *Any invariant group $\mathfrak{G}$ of ideal classes, and in particular $\mathfrak{C}$ itself, is the direct product of elementary subgroups.*

A 2. *Each elementary subgroup is of prime power order p^r.*

A 3. *If the order p^r of the elementary subgroup $\mathfrak{E}$ and the degree $(K:k)$ are relatively prime, then $\mathfrak{E}$ is of type $(p^s,..., p^s)$, with constant s. The only invariant subgroups of $\mathfrak{E}$ are, in obvious notation, $\mathfrak{E}^{p^\nu}$ $(\nu = 0, 1,..., s)$. $\mathfrak{E}$ and its invariant subgroups are indecomposable, i.e. neither $\mathfrak{E}$ nor any of its invariant subgroups can be written as a direct product of two proper subgroups one of which is invariant.*

A 4. *Those classes C of an invariant group $\mathfrak{G}$ which contain ideals in a field K', the invariant field of a normal subgroup Ω of Δ, form an invariant subgroup $\mathfrak{G}(K')$ of $\mathfrak{G}$.*

The subgroup of $\mathfrak{C}$ consisting of all ideal classes whose order in $\mathfrak{C}$ is relatively prime to $(K:k)$ will be called the *first group* of $\mathfrak{C}$ and denoted by $\mathfrak{C}_1$. Then $\mathfrak{C} = \mathfrak{C}_1 \times \mathfrak{C}_2$. $\mathfrak{C}_2$ will be called the *second group* of $\mathfrak{C}$. Both $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are invariant. If h is the *class number*, i.e. the order of $\mathfrak{C}$, and h_i the order of $\mathfrak{C}_i$ $(i = 1, 2)$, then $h = h_1 h_2$,

$$(h_1, (K:k)) = 1, \qquad (h, (K:k)) = (h_2, (K:k)).$$

An elementary group $\mathfrak{E} \neq \{I\}$ is a subgroup of either $\mathfrak{C}_1$ or $\mathfrak{C}_2$.

If a subgroup Σ of Δ leaves an invariant group $\mathfrak{G}$ of ideal classes fixed, then so do all groups conjugate to Σ in Δ; if two groups Σ and Σ' leave fixed, then so does the group $\{\Sigma, \Sigma'\}$ generated by Σ and Σ'. Hence:

B 1. *For every invariant group $\mathfrak{G}$ of ideal classes there exists a unique subgroup $\Sigma(\mathfrak{G})$ of Δ, which leaves $\mathfrak{G}$ fixed and contains all subgroups of Δ leaving $\mathfrak{G}$ fixed. $\Sigma(\mathfrak{G})$ is a normal subgroup of Δ.*

Assume that a class $C \neq I$ in an elementary group $\mathfrak{E}$ is invariant under a normal subgroup Ω of Δ. Then Ω leaves fixed the invariant subgroup $\{C\}$ of $\mathfrak{E}$ which is generated by C and all its conjugates under Δ. If $\mathfrak{E} \subset \mathfrak{C}_1$, then, by A 3, $\{C\} = \mathfrak{E}^{p^\nu}$, $\nu < s$. Since the orders of $\mathfrak{E}$ and Δ are then relatively prime, it follows, as in (**3**), Theorem 5.1, that Ω leaves $\mathfrak{E}$ fixed. Thus we obtain:

B 2. *If a normal subgroup Ω of Δ leaves one class $C \neq I$ of an elementary subgroup $\mathfrak{E} \subset \mathfrak{C}_1$ fixed, then it leaves $\mathfrak{E}$ fixed.*

Let Ω be again a normal subgroup of Δ. If, for a class C in an elementary subgroup $\mathfrak{E} \subset \mathfrak{C}_1$, $C^{N(\Omega)} \neq I$, then Ω leaves $C^{N(\Omega)}$ invariant, and therefore leaves $\mathfrak{E}$ fixed. If Ω has order m, and $\mathfrak{E}$ has exponent p^s, where the *exponent* of a group is defined as the lowest common multiple of the element-orders, then $(m, p^s) = 1$; hence

$$\exists\, m' \ . \ mm' \equiv 1 \pmod{p^s}.$$

Thus $C = C^{m'm} = C^{m'N(\Omega)}$ for all $C \in \mathfrak{E}$. We can therefore state:

B 3. *If Ω is a normal subgroup of Δ, and $\mathfrak{E}$ an elementary subgroup of $\mathfrak{C}_1$, then, in obvious notation, either $\mathfrak{E}^{N(\Omega)} = \mathfrak{E}$, or $\mathfrak{E}^{N(\Omega)} = \{I\}$. The former is true if and only if Ω leaves $\mathfrak{E}$ fixed.*

If the normal subgroup Ω of Δ leaves the invariant group $\mathfrak{G} \subset \mathfrak{C}_1$ fixed, then it follows from A 1 and B 3 that every class $C \in \mathfrak{G}$ is of the form $C = C'^{N(\Omega)}$. Hence C contains ideals of the invariant field K_Ω of Ω. Conversely, if every class $C \in \mathfrak{G}$ contains ideals in K_Ω, then Ω leaves $\mathfrak{G}$ fixed. Hence:

B 4. *A normal subgroup Ω of Δ leaves an invariant group $\mathfrak{G} \subset \mathfrak{C}_1$ fixed if and only if every class in $\mathfrak{G}$ contains ideals in the invariant field K_Ω of Ω.*

If $\mathfrak{G}$ is an invariant subgroup of $\mathfrak{C}_1$, Ω a normal subgroup of Δ, and $\mathfrak{G}(K_\Omega)$ the invariant subgroup of $\mathfrak{G}$ as defined in A 4, then it follows from B 2 and B 4 that for every elementary subgroup $\mathfrak{E}$ of $\mathfrak{G}$ either

$$\mathfrak{E} \wedge \mathfrak{G}(K_\Omega) = \mathfrak{E} \quad \text{or} \quad \mathfrak{E} \wedge \mathfrak{G}(K_\Omega) = \{I\}.$$

Let $\mathfrak{E}_i$ $(i = 1,..., r+t)$ be a set of disjoint elementary subgroups of $\mathfrak{G}$; $\mathfrak{E}_i \wedge \mathfrak{G}(K_\Omega) = \mathfrak{E}_i\,(i \leqslant r)$; $\mathfrak{E}_i \wedge \mathfrak{G}(K_\Omega) = \{I\}\,(i > r)$. Every class $C \in \prod_i \mathfrak{E}_i$ is then uniquely representable in the form $C = \prod_i C_i\,(C_i \in \mathfrak{E}_i)$ and $C^\omega = C$ if and only if for each 'component' C_i of C $C_i^\omega = C_i$, i.e. if and only if $C_i = I$ $(i > r)$. Hence $\left(\prod_i \mathfrak{E}_i\right) \wedge \mathfrak{G}(K_\Omega) = \prod_i \{\mathfrak{E}_i \wedge \mathfrak{G}(K_\Omega)\}$. Using A 1 we obtain:

B 5. *If Ω is a normal subgroup of Δ and K_Ω its invariant field, then any invariant subgroup $\mathfrak{G} \subset \mathfrak{C}_1$ can be written as a direct product $\mathfrak{G} = \mathfrak{G}(K_\Omega) \times \tilde{\mathfrak{G}}$, where $\mathfrak{G}(K_\Omega)$ and $\tilde{\mathfrak{G}}$ are invariant. Also $\mathfrak{G}(K_\Omega) = \mathfrak{G}^{N(\Omega)}$ and $\tilde{\mathfrak{G}} = \mathfrak{G}_{N(\Omega)}$, the subgroup of $\mathfrak{G}$ consisting of all classes $C \in \mathfrak{G}$ such that $C^{N(\Omega)} = I$.*

A group Δ, all of whose subgroups are normal, is called *Hamiltonian*; I shall also call the field K with Galois group Δ 'Hamiltonian'.

From B 1, B 2, and B 4 we now obtain:

B 6. *If K is Hamiltonian over k, then there exists for every elementary group $\mathfrak{E} \subset \mathfrak{C}_1$ a unique field K' between K and k such that every class $C \in \mathfrak{E}$ contains ideals in K', but, if $C \neq I$, C does not contain ideals in any field K'' such that $K \supset K'' \not\supset K'$.*†

If Ω, Ω' are two normal subgroups of Δ, and if $\mathfrak{G}$ is an invariant group in $\mathfrak{C}_1$, then it follows from B 4 that

$$\mathfrak{G}(K_\Omega) \wedge \mathfrak{G}(K_{\Omega'}) = \mathfrak{G}(K_{\Omega\Omega'}) = \mathfrak{G}(K_\Omega \wedge K_{\Omega'}).$$

† The symbol '$\supset$' is taken in the sense that '$=$' implies '$\supset$'.

From B 5 we then get

$$\mathfrak{G} = \tilde{\mathfrak{G}} \times \mathfrak{G}^*(K_\Omega) \times \mathfrak{G}^*(K_{\Omega'}) \times \mathfrak{G}(K_\Omega \wedge K_{\Omega'}),$$

where (i) no class $C \neq I$ in $\tilde{\mathfrak{G}}$ can be written as a product of classes containing ideals in K_Ω or $K_{\Omega'}$; (ii) every class C in $\mathfrak{G}^*(K_\Omega)$, $[\mathfrak{G}^*(K_{\Omega'})]$ contains ideals in K_Ω, $[K_{\Omega'}]$, but, if $C \neq I$, C does not contain ideals in $K_\Omega \wedge K_{\Omega'}$; (iii) $\mathfrak{G}(K_\Omega \wedge K_{\Omega'})$ is defined as in A 4. By repeated application of this decomposition process we obtain

B 7. *Let K be Hamiltonian over k, $\mathfrak{G}$ an invariant subgroup of $\mathfrak{C}_1$, e.g. $\mathfrak{C}_1$ itself. Then*

$$\mathfrak{G} = \dot{\prod_{K'}} \mathfrak{G}^*(K'), \qquad k \subset K' \subset K \tag{2.1}$$

where $\dot{\prod}$ denotes, as always throughout this work, the direct product, K' running through all fields between K and k (K and k included).

Every class $C \in \mathfrak{G}^*(K')$ contains ideals in K', but, if $C \neq I$, C cannot be written as the product of classes containing ideals in proper subfields of K' over k, and in particular C does not contain ideals of any field K'' between K and k, $K'' \not\supset K'$. The group $\mathfrak{G}(K'')$, as defined in A 4, *can be decomposed in the form*

$$\mathfrak{G}(K'') = \dot{\prod_{K'}} \mathfrak{G}^*(K'), \qquad k \subset K' \subset K''. \tag{2.2}$$

3. The first group of the class group

From now on the field K will be assumed to be Abelian over the base field k.

THEOREM 3.1. (Decomposition Theorem.) *Let K be an algebraic number field, Abelian over a field k. Let $\mathfrak{H}$ be an ideal group in K, invariant under the Galois group Δ of K over k, and $\mathfrak{C}_1$ be the first group of $\mathfrak{C} = \mathfrak{A}/\mathfrak{H}$. Then every invariant subgroup $\mathfrak{G}$ of $\mathfrak{C}_1$, and in particular $\mathfrak{C}_1$ itself, can be decomposed into a direct product of uniquely determined invariant subgroups*

$$\mathfrak{G} = \dot{\prod_{i}} \mathfrak{G}^*(K^{(i)}), \tag{3.1}$$

where (i) *$K^{(i)}$ runs through those subfields of K only, which are cyclic over k (k included),* (ii) *every class $C \in \mathfrak{G}^*(K^{(i)})$ contains ideals in $K^{(i)}$, but, if $C \neq I$, C cannot be written as a product of ideal classes containing ideals in proper subfields of $K^{(i)}$, and in particular C does not contain ideals in any field K', such that $K' \not\supset K^{(i)}$.*

If $K \supset K' \supset k$, $\mathfrak{G}$ can be decomposed in the form

$$\mathfrak{G} = \tilde{\mathfrak{G}} \times \mathfrak{G}(K'), \tag{3.2}$$

where $\mathfrak{G}(K')$ *consists of all those classes in* $\mathfrak{G}$ *which contain ideals in* K', *and*

$$\mathfrak{G}(K') = \dot{\prod_i} \mathfrak{G}^*(K^{(i)}) \quad (K^{(i)} \subset K'). \tag{3.3}$$

Proof. K is certainly Hamiltonian over k. The theorem thus follows from B 7 (§ 2), if it is shown that $\mathfrak{G}^*(K') = \{I\}$ if K' is not cyclic over k.

If $\mathfrak{G} = \dot{\prod_i} \mathfrak{E}_i$, where the $\mathfrak{E}_i$ are elementary groups, then

$$\mathfrak{G}^*(K') = \dot{\prod_i} \mathfrak{E}_i^*(K').$$

If the exponent of an elementary group $\mathfrak{E}$ is p^s, we conclude from B 2 that $\mathfrak{E}^*(K') = \{I\}$ if and only if $\mathfrak{E}^{p^{s-1}*}(K') = \{I\}$. We thus have only to prove that, if $\mathfrak{E}$ is an elementary group of exponent p, and p is relatively prime to the order of Δ, then a subgroup Ω of Δ with cyclic quotient group Δ/Ω leaves $\mathfrak{E}$ fixed.

Assume that $\mathfrak{E}$ is such a group. Then it is the representation module of an irreducible representation of Δ, and thus of the group ring $[\Delta]$ with coefficients in $GF(p)$. Δ is Abelian; hence by Schur's lemma the representation of $[\Delta]$ is isomorphic to $GF(p^m)$. Thus its multiplicative group is cyclic, and therefore a subgroup Ω of Δ with cyclic quotient group Δ/Ω is represented by the identical automorphism, i.e. leaves $\mathfrak{E}$ fixed.

Remarks on Theorem 3.1. (1) A weaker form of the theorem is given by the equation

$$\mathfrak{G} = \prod_i \mathfrak{G}(K^{(i)}) \quad (K^{(i)} \text{ cyclic over } k).$$

In particular, $\mathfrak{C}_1$ is generated by ideals of the unit class I and ideals in cyclic subfields of K.†

(2) A particular case of special interest is: K is *absolutely Abelian*, i.e. Abelian over the field of rationals, which we shall henceforth always denote by P; $\mathfrak{C}$ is the *absolute class group*, i.e. $\mathfrak{C} = \mathfrak{A}/\mathfrak{J}$, where throughout this work the term *absolute class group* is interpreted in the 'wider sense' given here. (We do not consider at all the class group *modulo* the totally positive principal ideals.)

In this case $\mathfrak{C}_1(\mathrm{P}) = \{I\}$, and thus the theorem remains true if $K^{(i)}$ runs through all proper cyclic extension fields of P only.

I shall now introduce the operations of *norm* and *transfer* on class groups. They will be defined here in a less general way than is usual.

Let K be Abelian over k, and K' a field between K and k. Let $\mathfrak{H}$ be

† A group $\mathfrak{G}$ of ideal classes is said to be *generated* by a set of ideals if, for every class $C \in \mathfrak{G}$, all ideals in C can be written as a product of ideals in the set.

an ideal group in K, and $\mathfrak{H}'$ an ideal group in K', both invariant under the Galois group Δ of K over k, such that the norms in K' of all ideals in $\mathfrak{H}$ fall into $\mathfrak{H}'$, and all ideals in $\mathfrak{H}'$ also lie in $\mathfrak{H}$. It is then easily seen that, (i) to every class C in $\mathfrak{C} = \mathfrak{A}/\mathfrak{H}$ there corresponds a unique class C' in $\mathfrak{C}' = \mathfrak{A}'/\mathfrak{H}'$ such that all norms of ideals in C fall into C', (ii) all ideals in a class C' of $\mathfrak{C}'$ lie in a unique class C of $\mathfrak{C}$. In the first case I call C' the *norm* of C, and denote it by $C' = \mathrm{N}_{K/K'}(C)$; in the second case I call C the *transfer* of C', and write $C = \mathrm{T}_{K/K'}(C')$. Wherever these concepts are used the groups $\mathfrak{H}$ and $\mathfrak{H}'$ have of course to be defined beforehand.

The mappings: $C \to \mathrm{N}_{K/K'}(C)$ and $C' \to \mathrm{T}_{K/K'}(C')$ are operator homomorphisms, with respect to Δ, of $\mathfrak{C}$ onto a subgroup $\mathrm{N}_{K/K'}(\mathfrak{C})$ of $\mathfrak{C}'$ in the first case, and of $\mathfrak{C}'$ onto a subgroup $\mathrm{T}_{K/K'}(\mathfrak{C}')$ of $\mathfrak{C}$ in the second case. If Ω is the Galois group of K over K' of order n, then

$$\mathrm{T}_{K/K'}\mathrm{N}_{K/K'}(C) = C^{N(\Omega)}, \qquad \mathrm{N}_{K/K'}\mathrm{T}_{K/K'}(C') = C'^{n}. \tag{3.4}$$

We now consider for each field K' between K and k (K, k included) the groups $\mathfrak{H}'$, generated by the principal ideals and the ideals in k. These groups satisfy the conditions for the definition of norm and transfer. If $k = \mathrm{P}$, then $\mathfrak{H}' = \mathfrak{I}'$. If for each field K', $\mathfrak{C}'$ is defined as $\mathfrak{A}'/\mathfrak{H}'$, then $\mathfrak{C}'^*(k) = \{I'\}$.

Let now $(K:k) = l^\mu$. Then $(K:K') = l^\nu\,(\nu \leqslant \mu)$. By (3.4), $\mathrm{T}_{K/K'}(C') = I$ implies $C'^{l^\nu} = I'$. Hence, if $C' \in \mathfrak{C}_1'$ and $\mathrm{T}_{K/K'}(C') = I$, then $C' = I'$. Therefore $\mathfrak{C}_1' \simeq \mathrm{T}_{K/K'}(\mathfrak{C}_1')$. But $\mathrm{T}_{K/K'}(\mathfrak{C}_1') = \mathfrak{C}_1(K')$, as defined in A 4 (§ 2). We thus have derived

Theorem 3.2. (Isomorphism Theorem.) *Let K be an algebraic number field, Abelian of prime power degree l^μ over a base field k, and let K' be a field between K and k; $\mathfrak{C}$ and $\mathfrak{C}'$ denote the class groups in K and K'* modulo *the groups generated by the ideals in k and the principal ideals in K and K' respectively, and $\mathfrak{C}_1$ and $\mathfrak{C}_1'$ their first groups. If $\mathfrak{C}_1(K')$ is the subgroup of $\mathfrak{C}_1$, consisting of all classes with ideals in K', then*

$$\mathfrak{C}_1(K') \simeq \mathfrak{C}_1'. \tag{3.5}$$

This isomorphism is an operator isomorphism with respect to the Galois group of K over k.

If K'' is a subfield of K' over k, then also

$$\mathfrak{C}_1(K'') \simeq \mathfrak{C}_1'(K''). \tag{3.6}$$

Corollary. *Let $K^{(i)}$ run through all the cyclic fields over k in K. Let h_1 be the order of $\mathfrak{C}_1$, and $h_1^{(i)}$ the order of $\mathfrak{C}_1^{(i)}$. If $K^{(i)}$ is contained in exactly r_i fields in K which are cyclic over k and of relative degree l over $K^{(i)}$, then*

$$h_1 = \prod_i (h_1^{(i)})^{1-r_i}. \tag{3.7}$$

If h, $h^{(i)}$ are the class numbers of K, $K^{(i)}$ respectively, then

$$h = l^{\lambda} \prod_i (h^{(i)})^{1-r_i}. \tag{3.7 a}$$

Proof. Let $g_1^{(i)}$ be the order of $\mathfrak{C}_1^*(K^{(i)})$, as defined in Theorem 3.1. $K^{(i)}$ $(\neq k)$ has a unique maximal proper subfield $K^{(i')}$; for $K^{(i')} = k$ we observe that $h_1^{(i')} = g_1^{(i')} = 1$. Hence, by Theorem 3.2,

$$g_1^{(i)} = \{\mathfrak{C}_1(K^{(i)}):\mathfrak{C}_1(K^{(i')})\} = \{\mathfrak{C}_1^{(i)}:\mathfrak{C}_1^{(i)}(K^{(i')})\} = h_1^{(i)}/h_1^{(i')}.$$

By Theorem 3.1, $h_1 = \prod_i g_1^{(i)}$. Hence

$$h_1 = \prod_i (h_1^{(i)}/h_1^{(i')}) = \prod_i h_1^{(i)}/\prod_i h_1^{(i')}.$$

For a fixed j, $h_1^{(j)}$ occurs as often in the denominator as $K^{(j)}$ occurs as a maximal proper subfield of cyclic fields in K, i.e. r_j times. This proves (3.7) and hence (3.7 a).

If in particular the Galois group of K over k has exponent l, then

$$h_1 = \prod_i h_1^{(i)}. \tag{3.7 b}$$

All these results hold in particular if K is absolutely Abelian and $\mathfrak{C}$ the absolute class group.

The number of elements in a basis of $\mathfrak{G}/\mathfrak{G}^p$ will be called the *p-dimension* of $\mathfrak{G}$ and denoted by $d_p(\mathfrak{G})$, or, if no misunderstanding is possible, by $d(\mathfrak{G})$.

Let K be Abelian of exponent l^ν (i.e. the Galois group Δ of exponent l^ν). Let $\mathfrak{C}$ be again the class group *modulo* the group generated by principal ideals and ideals in k. A subgroup $\mathfrak{E}$ of $\mathfrak{C}$ of exponent p^s $(p, l) = 1$ is, by A 3 (§ 2), elementary if and only if $\mathfrak{E}^{p^{s-1}}$ is irreducible with respect to Δ. Such an elementary group $\mathfrak{E}^{p^{s-1}}$ is (as seen in the proof of Theorem 3.1) representation module for a faithful, irreducible representation of a cyclic group $\Phi = \Delta/\Sigma(\mathfrak{E})$. The order of Φ is l^μ $(\mu \leqslant \nu)$. The corresponding representation of the group ring $[\Phi]$ with coefficients in $GF(p)$ is isomorphic to the field $GF(p^{m'})$ which is the field of l^μth roots of unity over $GF(p)$; m' is the order of p in the group $\mathfrak{R}(l^\mu)$ of prime-residue-classes mod l^μ. Hence the p-dimension of $\mathfrak{E}$ is m'.

If m is the order of p in $\mathfrak{R}(l)$, and $p^m \equiv 1+al^\lambda \pmod{l^{\lambda+1}}$, then ml^κ is the order of p in $\mathfrak{R}(l^{\lambda+\kappa})$. Writing $n =$ maximum $(\nu-\lambda, 0)$, we conclude that an elementary group of exponent p^s has p-dimensions $m \,.\, l^k$ $(0 \leqslant k \leqslant n)$.

A series of subgroups $\mathfrak{C} = \mathfrak{G}_0, \mathfrak{G}_1, \mathfrak{G}_2,\dots, \mathfrak{G}_t = \{I\}$, such that $\mathfrak{G}_j/\mathfrak{G}_{j+1}$ $(j = 0, 1, 2,\dots, t-1)$ is elementary, will be called an *invariant series* of $\mathfrak{C}$. Such series evidently exist, and the Jordan–Hölder theorem applies to

them. Thus the p_i-length, i.e. the number of factors $\mathfrak{G}_j/\mathfrak{G}_{j+1}$ whose order is a power of p_i, is defined for any such series, and is the same for all invariant series of $\mathfrak{C}$. Using A 1 (§ 2) we now obtain

THEOREM 3.3. (Dimension Theorem.) *Let K be an algebraic number-field, Abelian and of exponent l^ν over a base field k. For $0 < \rho \leqslant \nu$ let K_ρ be its maximal subfield of exponent l^ρ. If h denotes the class number of K* modulo *the group generated by the ideals in k and the principal ideals in K, then*

$$h = l^t \prod p_i^{m_i[a_{0i}+a_{1i}l+\ldots+a_{n_i i}l^{n_i}]}, \tag{3.8}$$

where (i) *the product extends over all primes $p_i \neq l$;*

(ii)
$$m_i = \text{minimum } r > 0, \qquad p_i^r \equiv 1 \pmod{l}, \tag{3.9}$$
$$n_i = \text{minimum } s \geqslant 0, \qquad p_i^{m_i} \equiv 1 \pmod{l^{\nu-s}}; \tag{3.10}$$

(iii) *the a_{ji} are integers ($\geqslant 0$), and, for $0 \leqslant \nu_j \leqslant n_i$, $a_{0i}+a_{1i}+\ldots+a_{\nu_i i}$ is the p_i-length of the invariant series of the class group in $K_{\nu-n_i+\nu_i}$,* modulo *the group generated by the ideals in k and the principal ideals in $K_{\nu-n_i+\nu_i}$.*

Theorem 3.3 imposes a restriction on the class number h of a relatively Abelian field K:

COROLLARIES. (1)
$$h = l^t \prod p_i^{c_i m_i}, \tag{3.11}$$

where the product extends over all primes $p_i \neq l$, the c_i are integers ($\geqslant 0$), *and the m_i are determined as in* (3.9).

(2) *If $\rho \geqslant \nu-n_i$, and if h_ρ denotes the class number of the field K_ρ,* modulo *the group generated by the ideals in k and the principal ideals in K_ρ, then*

$$h/h_\rho = q_i p_i^{c_i' m_i l^{\rho-\nu+n_i+1}}, \tag{3.12}$$

where c_i' is an integer, $0 \leqslant c_i' \leqslant c_i/l^{\rho-\nu+n_i+1}$, and q_i is a rational number whose denominator and numerator are prime to p_i.

[*Note added* 25 *April* 1952.] I find now that a paper by H. Nehrkorn, 'Über absolute Idealklassengruppen und Einheiten in algebraischen Zahlkorpern' in *Abh. Math. Sem. Hamburg*, 1933, is relevant to the subject-matter of the present paper. His results overlap partially with some of mine.

REFERENCES

1. H. Hasse, 'Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper. Teil I: Klassenkörpertheorie.' *Jahresberichte der Deutscher Math. Vereinigung*, 35 (1926).
   Teil Ia: Beweise zu Teil I. Ibid. 36 (1927).
   Teil II: Reziprozitäts–Gesetz. Ibid., Ergänzungsband VI (1930).
2. B. L. van der Waerden, *Moderne Algebra* (Berlin, 1937).
3. A. Fröhlich, 'The representation of a finite group as a group of automorphisms on a finite Abelian group', *Quart. J. of Math.* (Oxford) (2) 1 (1950), 270–83.

LAMÉ-WANGERIN FUNCTIONS

By C. G. LAMBE (*Shrivenham*)

[Received 5 April 1951]

Summary

In this paper the products of solutions of the Lamé–Wangerin differential equation are expressed in terms of the solutions of the equation obtained by Halphen's transformation, and expressions for the characteristic solutions in finite form are derived. An integral equation is given which is satisfied by these solutions and is of the same form as that for Lamé functions.

1. Introduction

Lamé–Wangerin functions are the characteristic solutions of Lamé's differential equation

$$\frac{d^2\lambda}{du^2} = \{n(n+1)\wp(u)+h\}\lambda$$

when n is half of an odd integer.

Wangerin (**1**, **2**) derived the form of the differential equation from his studies of orthogonal surfaces and Haentzschel (**3**) gave some theoretical results for the integral of the differential equation and its connexion with special functions which are particular cases. Halphen (**4**) considered the differential equation obtained by the transformation $u = 2v$ and showed that the product of three solutions of order $\frac{1}{2}$ can represent a steady state of temperature which would be represented by an infinite series of Lamé functions.

Brioschi (**5**, **6**, **7**) derived the form of the equation by considering possible types of equation the product of whose solutions squared should be a polynomial. He developed expressions for these products of solutions and deduced expressions for the fundamental solutions. Crawford (**8**) showed that for real values of the constants of the differential equation the characteristic values of h are real and distinct.

The characteristic solutions of the differential equation are used in problems of potential theory applied to confocal cyclides or their degenerate forms. The non-characteristic solutions were used by Poole (**9**) to establish Dirichlet's principle for a flat ring. An interesting feature of the Lamé–Wangerin equation is that a complete solution of the differential equation can be expressed in terms of the characteristic functions.

2. The transformed equation

Lamé's equation in its algebraic form is

$$4x(x-1)(x-a)\frac{d^2u}{dx^2}+2\{3x^2-2(a+1)x+a\}\frac{du}{dx}-\{n(n+1)x+h\}u = 0, \tag{2.1}$$

and there is no loss of generality in taking the finite singularities as 0, 1, and a. I shall assume a real and greater than unity and $n = m+\frac{1}{2}$ where m is a positive integer or zero.

Putting

$$x = \frac{(z^2-a)^2}{4z(z-1)(z-a)} \quad \text{and} \quad u(x) = \{z(z-1)(z-a)\}^{-n/2}v(z), \tag{2.2}$$

we have the transformed equation considered by Halphen and others [see, for example, (**10**), examples 11 and 12]

$$z(z-1)(z-a)\frac{d^2v}{dz^2}-m\{3z^2-2(a+1)z+a\}\frac{dv}{dz}+$$
$$+\{m(2m+1)z-h-(m+\tfrac{1}{2})^2(a+1)\}v = 0. \tag{2.3}$$

If h has one of a set of $m+1$ characteristic values, this equation has two solutions which are polynomials in z, namely

$$v_1(z) = \sum_{r=0}^{m} c_r a^{m+\frac{1}{2}-r}z^r, \qquad v_2(z) = \sum_{r=0}^{m} c_r z^{2m+1-r}, \tag{2.4}$$

the coefficients c_r being the same in the two solutions and $c_0 = 1$. Substitution of either of these solutions in (2.3) leads to the recurrence relations

$$(r-m)(r+1)c_{r+1}-\{h+(m+\tfrac{1}{2}-r)^2(a+1)\}c_r+$$
$$+a(m+1-r)(2m+2-r)c_{r-1} = 0 \tag{2.5}$$

for $r = 0, 1,..., m$.

The first m relations determine the coefficients c_1 to c_m, the coefficient c_r being a polynomial of degree r in h. Putting $r = m$ we have

$$-\{h+\tfrac{1}{4}(a+1)\}c_m+(m+2)ac_{m-1} = 0. \tag{2.6}$$

The expression in (2.6) is a polynomial of degree $m+1$ in h, determining $m+1$ characteristic values of h, which are real and distinct. There are thus $2m+2$ ($= 2n+1$) characteristic solutions of (2.3) which are linearly independent.

We have also

$$v_2(z) = \frac{z^{2m+1}}{a^{m+\frac{1}{2}}}v_1\left(\frac{a}{z}\right), \qquad v_1(z) = \frac{z^{2m+1}}{a^{m+\frac{1}{2}}}v_2\left(\frac{a}{z}\right). \tag{2.7}$$

3. Products of solutions

The differential equation satisfied by the squares and the product of solutions of Lamé's equation is **(10)**

$$x(x-1)(x-a)\frac{d^3X}{dx^3}+\tfrac{3}{2}\{3x^2-2(a+1)x+a\}\frac{d^2X}{dx^2}+$$

$$+\{(3-n-n^2)x-(h+a+1)\}\frac{dX}{dx}-\tfrac{1}{2}n(n+1)X = 0. \quad (3.1)$$

Taking $n = m+\frac{1}{2}$ and writing a solution as

$$L(x) = \sum_{r=0}^{m} b_r x^{m+\frac{1}{2}-r} \quad (3.2)$$

we have on substituting in (3.1) the recurrence relations

$$(r+1)(r-m)(2m+1-r)b_{r+1}-$$
$$-(m+\tfrac{1}{2}-r)\{h+(m+\tfrac{1}{2}-r)^2(a+1)\}b_r+$$
$$+a(m+\tfrac{3}{2}-r)(m+1-r)(m+\tfrac{1}{2}-r)b_{r-1} = 0 \quad (3.3)$$

for $r = 0, 1,..., m$.

With $b_0 = 1$, the recurrence relations determine the coefficients b_1 to b_m, and the characteristic equation for h is

$$-\{h+\tfrac{1}{4}(a+1)\}b_m+\tfrac{3}{2}ab_{m-1} = 0, \quad (3.4)$$

giving, as will appear later, the same values of h as (2.6).

The equation (3.1) may be integrated once, giving

$$x(x-1)(x-a)\{2XX''-X'^2\}+\{3x^2-2(a+1)x+a\}XX'-$$
$$-\{n(n+1)x+h\}X^2 = C, \quad (3.5)$$

where C is a constant. If C is taken as zero, the substitution $X = u^2$ gives the equation (2.1). Hence the product (3.2) is a particular integral of (3.5) with a non-zero constant.

If $$X = x^{\frac{1}{2}}(b_m+b_{m-1}x+...),$$

then, when $x = 0$,

$$XX' = \tfrac{1}{2}b_m^2, \qquad x(2XX''-X'^2) = -\tfrac{3}{4}b_m^2,$$

and hence $$C = -\tfrac{1}{4}ab_m^2.$$

If the roots of (3.2) be α_r, that is

$$L(x) = x^{\frac{1}{2}}\prod_{r=1}^{m}(x-\alpha_r),$$

then $$C = -\tfrac{1}{4}a\prod_{r=1}^{m}\alpha_r^2.$$

Again, putting $x = \alpha_r$ in (3.5), since

$$L'(\alpha_r) = \alpha_r^{\frac{1}{2}} \prod_s{}' (\alpha_r-\alpha_s),$$

we have $$\tfrac{1}{4}a \prod_r \alpha_r^2 = \alpha_r^2(\alpha_r-1)(\alpha_r-a) \prod_s{}' (\alpha_r-\alpha_s)^2,$$

that is $$\prod_s{}' \left(\frac{\alpha_s}{\alpha_r-\alpha_s}\right)^2 = 4(\alpha_r-1)(\alpha_r-a)/a \tag{3.6}$$

for $r = 1, 2,..., m$.

It follows that the roots must be distinct and not zero, and that there can be no real roots between 1 and a.

Taking the square roots of the expressions in (3.6) we have

$$2(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}} = \pm a^{\frac{1}{2}} \prod_s{}' \left(\frac{\alpha_s}{\alpha_r-\alpha_s}\right),$$

and hence, if the surds are taken with appropriate signs

$$\sum_{r=1}^{m} (\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}} = \tfrac{1}{2}a^{\frac{1}{2}}. \tag{3.7}$$

The roots α_r may be complex. For example, if $m = 2$, there is a characteristic value of h which gives complex roots.

The characteristic solutions of (2.3) may be considered as polynomials in z, $z-1$, or $z-a$, and hence from the symmetry of the transformation of z to x it is evident that the equation (3.1) has also solutions of the form

$$M(x) = \sum_{r=0}^{m} d_r(x-1)^{m+\frac{1}{2}-r}, \qquad N(x) = \sum_{r=0}^{m} e_r(x-a)^{m+\frac{1}{2}-r}, \tag{3.8}$$

and the expressions $L(x)$, $M(x)$, $N(x)$ form a fundamental set of solutions of (3.1).

4. Solutions of the transformed equation

Substituting $$x = \frac{(z^2-a)^2}{4z(z-1)(z-a)}$$

we have

$$\{4z(z-1)(z-a)\}^n L(x) = (z^2-a) \prod_{r=1}^{m} \{(z^2-a)^2-4\alpha_r z(z-1)(z-a)\}.$$

Hence the product of corresponding solutions of the transformed equation must be

$$V(z) = (z^2-a) \prod_{r=1}^{m} \{(z^2-a)^2-4\alpha_r z(z-1)(z-a)\}. \tag{4.1}$$

Now $v_2(z)-v_1(z)$ has the factor $(z-a^{\frac{1}{2}})$, and, if β is a root, a/β is also a root from (2.7). Hence we may write

$$v_2(z)-v_1(z) = (z-a^{\frac{1}{2}}) \prod_{r=1}^{m} (z^2-2p_r z+a) \tag{4.2}$$

and similarly

$$v_2(z)+v_1(z) = (z+a^{\frac{1}{2}}) \prod_{r=1}^{m} (z^2-2q_r z+a). \tag{4.3}$$

That these are the only solutions containing the factors $(z-a^{\frac{1}{2}})$ and $(z+a^{\frac{1}{2}})$ respectively follows from the fact that $v_2(z)$ and $v_1(z)$ cannot have a common factor $(z-a^{\frac{1}{2}})$ or $(z+a^{\frac{1}{2}})$ without having a repeated root, and from the differential equation the only possible repeated roots are 0, 1, and a.

Hence we have $$V(z) = v_2^2(z)-v_1^2(z).$$

We have also $$2\sum_{r=1}^{m} p_r+a^{\frac{1}{2}} = 2\sum_{r=1}^{m} q_r-a^{\frac{1}{2}} = -c_1,$$

and hence $$\sum_{r=1}^{m} (q_r-p_r) = a^{\frac{1}{2}}. \tag{4.4}$$

When $x = \alpha_r$, $\quad z = \{\alpha_r^{\frac{1}{2}}\pm(\alpha_r-1)^{\frac{1}{2}}\}\{\alpha_r^{\frac{1}{2}}\pm(\alpha_r-a)^{\frac{1}{2}}\}$,

and hence sums of pairs of values z whose product is a are

$$2\{\alpha_r\pm(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}\}.$$

Therefore taking

$$p_r = \alpha_r-(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}$$

$$q_r = \alpha_r+(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}$$

the equation (4.4) is satisfied from (3.7), and we have

$$v_2(z)-v_1(z) = (z-a^{\frac{1}{2}}) \prod_{r=1}^{m} [z^2-2z\{\alpha_r-(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}\}+a],$$

$$v_2(z)+v_1(z) = (z+a^{\frac{1}{2}}) \prod_{r=1}^{m} [z^2-2z\{\alpha_r+(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}\}+a]. \tag{4.5}$$

5. Solutions of the Lamé–Wangerin equation

We have

$$(x-1)^{\frac{1}{2}} = \frac{z^2-2z+a}{\{4z(z-1)(z-a)\}^{\frac{1}{2}}},$$

$$(x-a)^{\frac{1}{2}} = \frac{z^2-2az+a}{\{4z(z-1)(z-a)\}^{\frac{1}{2}}}.$$

Therefore

$$a^{\frac{1}{2}}(x-1)^{\frac{1}{2}}+(x-a)^{\frac{1}{2}} = \frac{(a^{\frac{1}{2}}+1)(z-a^{\frac{1}{2}})^2}{\{4z(z-1)(z-a)\}^{\frac{1}{2}}},$$

$$a^{\frac{1}{2}}(x-1)^{\frac{1}{2}}-(x-a)^{\frac{1}{2}} = \frac{(a^{\frac{1}{2}}-1)(z+a^{\frac{1}{2}})^2}{\{4z(z-1)(z-a)\}^{\frac{1}{2}}}.$$

Also

$$\{(\alpha_r-a)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}\pm(\alpha_r-1)^{\frac{1}{2}}(x-a)^{\frac{1}{2}}\}\{4z(z-1)(z-a)\}^{\frac{1}{2}}$$
$$= \{(\alpha_r-a)^{\frac{1}{2}}\pm(\alpha_r-1)^{\frac{1}{2}}\}(z^2+a)-2z\{(\alpha_r-a)^{\frac{1}{2}}\pm a(\alpha_r-1)^{\frac{1}{2}}\}$$
$$= \{(\alpha_r-a)^{\frac{1}{2}}\pm(\alpha_r-1)^{\frac{1}{2}}\}\{z^2-2z[\alpha_r\mp(\alpha_r-1)^{\frac{1}{2}}(\alpha_r-a)^{\frac{1}{2}}]+a\}.$$

We may therefore take as the solutions of (2.1)

$$E_n(x) = (a-1)^{-\frac{1}{2}n}\{(a-x)^{\frac{1}{2}}+(a-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}\times$$
$$\times \prod_{r=1}^{m} \{(1-\alpha_r)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}+(a-\alpha_r)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\}, \quad (5.1)$$

$$F_n(x) = (a-1)^{-\frac{1}{2}n}\{(a-x)^{\frac{1}{2}}-(a-ax)^{\frac{1}{2}}\}^{\frac{1}{2}}\times$$
$$\times \prod_{r=1}^{m} \{(1-\alpha_r)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}-(a-\alpha_r)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\}, \quad (5.2)$$

and
$$E_n(x)F_n(x) = x^{\frac{1}{2}} \prod_{r=1}^{m} (x-\alpha_r).$$

In terms of Jacobian elliptic functions, writing $x = \mathrm{sn}^2 u$ with $\kappa^2 = 1/a$ we have

$$E_n(u) = (1-\kappa^2)^{-\frac{1}{2}n}(\mathrm{dn}\,u+\mathrm{cn}\,u)^{\frac{1}{2}} \prod_{r=1}^{m} (\mathrm{dn}\,u\,\mathrm{cn}\,\omega_r+\mathrm{cn}\,u\,\mathrm{dn}\,\omega_r), \quad (5.3)$$

$$F_n(u) = (1-\kappa^2)^{-\frac{1}{2}n}(\mathrm{dn}\,u-\mathrm{cn}\,u)^{\frac{1}{2}} \prod_{r=1}^{m} (\mathrm{dn}\,u\,\mathrm{cn}\,\omega_r-\mathrm{cn}\,u\,\mathrm{dn}\,\omega_r), \quad (5.4)$$

and
$$E_n(u)F_n(u) = \mathrm{sn}\,u \prod_{r=1}^{m} (\mathrm{sn}^2 u-\mathrm{sn}^2\omega_r). \quad (5.5)$$

These solutions are in agreement with more complicated forms obtained by Brioschi (7), namely

$$y_1 = \{L(x)\}^{\frac{1}{2}} \left\{\frac{t_1(0)}{t_2(0)}\right\}^{\frac{1}{4}} \prod_{r=1}^{m} \left\{\frac{t_1(\alpha_r)}{t_2(\alpha_r)}\right\}^{\frac{1}{2}},$$

$$y_2 = \{L(x)\}^{\frac{1}{2}} \left\{\frac{t_2(0)}{t_1(0)}\right\}^{\frac{1}{4}} \prod_{r=1}^{m} \left\{\frac{t_2(\alpha_r)}{t_1(\alpha_r)}\right\}^{\frac{1}{2}}, \quad (5.6)$$

where

$$t_1(\alpha) = (x-1)^{\frac{1}{2}}(x-a)^{\frac{1}{2}}-(\alpha-1)^{\frac{1}{2}}(\alpha-a)^{\frac{1}{2}}-2(x-\alpha),$$
$$t_2(\alpha) = (x-1)^{\frac{1}{2}}(x-a)^{\frac{1}{2}}+(\alpha-1)^{\frac{1}{2}}(\alpha-a)^{\frac{1}{2}}-2(x-\alpha).$$

6. Expressions for the products of solutions

In the recurrence relations (3.3) for $L(x)$ write†

$$b_r = \frac{\Gamma(2m+2-r)}{\Gamma(m+\frac{3}{2}-r)}\,c_r.$$

We then have

$$(r+1)(r-m)c_{r+1}-\{h+(m+\tfrac{1}{2}-r)^2(a+1)\}c_r+$$
$$+a(m+1-r)(2m+2-r)c_{r-1} = 0.$$

These are the same recurrence relations as (2.5) and $r = m$ leads to the same characteristic equation for h.

† It has been pointed out to me by a referee that this relation is one of 'transference' in the sense of Burchnall and Chaundy, e.g. *Proc. London Math. Soc.* 2 (50); (18) and (19).

Hence
$$L(x) = \sum_{r=0}^{m} \frac{\Gamma(2m+2-r)}{\Gamma(m+\frac{3}{2}-r)} c_r x^{m+\frac{1}{2}-r}. \tag{6.1}$$

Defining a fractional differential as
$$D^{m+\frac{1}{2}} x^p = \frac{\Gamma(p+1)}{\Gamma(p+\frac{1}{2}-m)} x^{p-m-\frac{1}{2}}$$
for integer values of p, we have
$$L(x) = \frac{\Gamma(m+\frac{3}{2})}{\Gamma(2m+2)} D^{m+\frac{1}{2}} \sum_{r=0}^{m} c_r x^{2m+1-r} = \frac{\Gamma(m+\frac{3}{2})}{\Gamma(2m+2)} D^{m+\frac{1}{2}} v_2(x). \tag{6.2}$$
We may express this equation in the form of an integral by using an integral relation given by the author **(11)**, namely,
$$v_2(x) = \lambda \int\limits_{(0+)} (x-t)^{2m+1}\{t(t-1)(t-a)\}^{-m-1} v_1(t)\, dt. \tag{6.3}$$
$$D^{m+\frac{1}{2}}(x-t)^{2m+1} = D^{m+\frac{1}{2}} \sum_{r=0}^{2m+1} \frac{\Gamma(2m+2)}{r!\,\Gamma(2m+2-r)} x^{2m+1-r}(-t)^r$$
$$= \frac{\Gamma(2m+2)}{\Gamma(m+\frac{3}{2})} \sum_{r=0}^{2m+1} \frac{\Gamma(m+\frac{3}{2})}{r!\,\Gamma(m+\frac{3}{2}-r)} x^{m+\frac{1}{2}-r}(-t)^r.$$
In the contour integral, the part of this expression containing powers of t greater than m will have zero residue at $t = 0$, and hence the value of the integral is unchanged if under the integral sign we take
$$D^{m+\frac{1}{2}}(x-t)^{2m+1} = \frac{\Gamma(2m+2)}{\Gamma(m+\frac{3}{2})} (x-t)^{m+\frac{1}{2}}.$$
Hence, using (6.2), we have
$$L(x) = \lambda_1 \int\limits_{(0+)} (x-t)^{m+\frac{1}{2}}\{t(t-1)(t-a)\}^{-m-1} v_1(t)\, dt. \tag{6.4}$$
Similarly, we have
$$M(x) = \lambda_2 \int\limits_{(1+)} (x-t)^{m+\frac{1}{2}}\{t(t-1)(t-a)\}^{-m-1} v_1(t)\, dt, \tag{6.5}$$
$$N(x) = \lambda_3 \int\limits_{(a+)} (x-t)^{m+\frac{1}{2}}\{t(t-1)(t-a)\}^{-m-1} v_1(t)\, dt. \tag{6.6}$$
The expressions for $M(x)$ and $N(x)$ follow by symmetry since $v_1(z)$ is a solution of (2.3) of degree m appropriate to any of the singularities $z = 0$, $z = 1$, $z = a$.

7. Integral equations

Integral equations satisfied by the solutions of the transformed Lamé–Wangerin equation have been given by the author in the reference quoted in the previous section.

For example, the integral equation

$$\phi(z) = \lambda \int\limits_{(1+,a+)} (z-t)^{2n}\{t(t-1)(t-a)\}^{-n-\frac{1}{2}}\phi(t)\,dt \tag{7.1}$$

has solutions $\phi(z) = v_2(z) \pm v_1(z)$.

Writing $\psi(z) = \{z(z-1)(z-a)\}^{-\frac{1}{2}n}\phi(z)$

we have

$$\psi(z) = \lambda \int\limits_{(1+,a+)} \frac{(z-t)^{2n}\psi(t)\,dt}{\{z(z-1)(z-a)t(t-1)(t-a)\}^{\frac{1}{2}n}\{t(t-1)(t-a)\}^{\frac{1}{2}}}, \tag{7.2}$$

of which the solutions are $E_n(x)$ or $F_n(x)$ expressed in terms of z.

The integral equation may be expressed in terms of Weierstrassian elliptic functions by transforming the singularities from 0, 1, a to e_1, e_2, e_3, and writing

$$x = \wp(u), \qquad z = \wp(\tfrac{1}{2}u), \qquad t = \wp(\tfrac{1}{2}v).$$

We have

$$\theta(u) = \mu \int\limits_{C} \frac{\{\wp(\frac{1}{2}u)-\wp(\frac{1}{2}v)\}^{2n}}{\{\wp'(u)\wp'(v)\}^{n}}\theta(v)\,dv, \tag{7.3}$$

which has as its solutions the $2n+1$ characteristic solutions of the Lamé–Wangerin equation

$$\frac{d^2\lambda}{du^2} = \{n(n+1)\wp(u)+h\}\lambda, \tag{7.4}$$

C being a contour encircling $v = \omega_1$ and $v = \omega_2$.

Now, when n is an integer, the $2n+1$ characteristic solutions of the transformed Lamé equation (2.3) satisfy the integral equation (7.1) [(**12**) equation (6.6)] and hence the $2n+1$ Lamé functions satisfy the integral equation (7.3). This integral equation is therefore satisfied by the $2n+1$ characteristic solutions of Lamé's equation (7.4) when n is an integer or half of an odd integer.

REFERENCES

1. A. Wangerin, *Berliner Monatsberichte*, Feb. 1878.
2. —— *Encyklopädie*, II, A 10, 740.
3. E. Haentzschel, *Studien über die Reduktion der Potentialgleichung* (Berlin, 1893).
4. G. H. Halphen, *Traité des Fonctions Elliptiques* (Paris, 1888), t. 2, 482.
5. F. Brioschi, *Comptes Rendus*, 86 (1878), 313.
6. —— *Annali di Mat.* (2) 11 (1878).
7. —— *Rend. R. Acad. dei Lincei*, (5) 1 (1892), 327.
8. L. Crawford, *Quart. J. of Math.* 27 (1895), 93.
9. E. G. C. Poole, *Proc. London Math. Soc.* (2) 29 (1929), 342; 30 (1930), 174.
10. E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1920), 23.
11. C. G. Lambe, *Proc. Edinburgh Math. Soc.* (appearing shortly).
12. —— *Quart. J. of Math.* (Oxford), (2) 5 (1951), 53.

ON THE BOUNDS OF A BILINEAR FORM RELATED TO HILBERT'S

By J. D. WESTON (*Newcastle upon Tyne*)

[Received 6 April 1951]

IN 1911 I. Schur† proved that, when λ is real and not an integer, and the numbers x_n and y_n are all positive, the bilinear form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x_m y_n}{m-n+\lambda}$$

is bounded above by $\pi\,|\operatorname{cosec}\pi\lambda|\,\sum_1^\infty x_n^2\sum_1^\infty y_n^2$. In this note I show that this is the exact upper bound of the modulus of the form, and that it is not attained. I also obtain a result on the bounds of the associated quadratic form, which requires a separate discussion owing to the lack of symmetry.

Let $\mathfrak{H}$ be the Hilbert space whose vectors x are doubly infinite sequences of complex numbers x_n such that $\sum_{-\infty}^{\infty}|x_n|^2<\infty$, with the scalar product $(x,y)=\sum_{-\infty}^{\infty}x_n\overline{y_n}$. A vector x will be called *real* if every x_n is real, and *positive* if every x_n is non-negative and $\|x\|>0$, where $\|x\|=(x,x)^{\frac{1}{2}}$. When the number λ is real and not an integer, U_λ will denote the linear transformation in $\mathfrak{H}$ represented by the matrix $(U_{\lambda;m,n})$, where, for all integers m and n,

$$U_{\lambda;m,n}=\frac{\sin\pi\lambda}{\pi(m-n+\lambda)}.$$

Thus

$$(x,U_\lambda y)=\sum_{m=-\infty}^{\infty}x_m\sum_{n=-\infty}^{\infty}U_{\lambda;m,n}\overline{y_n}=\frac{\sin\pi\lambda}{\pi}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{x_m\overline{y_n}}{m-n+\lambda}.$$

It is easy to verify (by the calculus of residues, for example) that the rows of the matrix $(U_{\lambda;m,n})$ are mutually orthogonal unit vectors in $\mathfrak{H}$. The same is true of the columns, so that U_λ is a unitary transformation.‡ Hence, if x and y are any vectors in $\mathfrak{H}$,

$$(x,U_\lambda y)\leqslant\|x\|\,\|U_\lambda y\|=\|x\|\,\|y\|,$$

equality occurring if and only if $U_\lambda y$ is proportional to x.

† *Journal für Math.* **140** (1911), 1–28.

‡ See, for example, M. H. Stone, *Linear Transformations in Hilbert Space* (American Math. Soc. Colloquium Publication, Vol. 15, 1932).

Quart. J. Math. Oxford (2), 3 (1952), 115–18.

Now let $\mathfrak{M}_p$ be defined, for each integer p, as the linear manifold in $\mathfrak{H}$ consisting of all those vectors x for which $x_n = 0$ whenever $n < p$. Then, if x and y are both in $\mathfrak{M}_1$,

$$(x, U_\lambda y) = \frac{\sin \pi\lambda}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m \overline{y_n}}{m-n+\lambda}.$$

Hence, if x and y are unit vectors $\left(\text{so that } \sum_1^\infty |x_n|^2 = \sum_1^\infty |y_n|^2 = 1\right)$,

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m \overline{y_n}}{m-n+\lambda} \right| \leqslant \pi\, |\operatorname{cosec} \pi\lambda|.$$

This extends Schur's result to complex vectors. Moreover, strict inequality holds if $\lambda > 0$ and x is positive, or if $\lambda < 0$ and y is positive. For $(x, U_\lambda y) = \overline{(y, U_{-\lambda} x)}$, and, if $\lambda > 0$ and x is a positive vector in $\mathfrak{M}_1$, it is evident that the vector $U_{-\lambda} x$ does not lie in $\mathfrak{M}_1$ and is therefore not proportional to y; hence in this case $|(y, U_{-\lambda} x)| < \|x\|\,\|y\|$. Similarly, if $\lambda < 0$ and y is positive, $|(x, U_\lambda y)| < \|x\|\,\|y\|$. In particular, if x and y are both positive unit vectors in $\mathfrak{M}_1$, $|(x, U_\lambda y)| < 1$ for every λ.

Next, if p and q are any integers and x and y are unit vectors in $\mathfrak{M}_p$, it is clear that the vectors x' and y' defined by the equations $x'_n = x_{n+p-q}$ and $y'_n = y_{n+p-q}$ are unit vectors in $\mathfrak{M}_q$ such that $(x, U_\lambda y) = (x', U_\lambda y')$. Hence the bounds of $(x, U_\lambda y)$ in $\mathfrak{M}_p$ (the upper and lower bounds of $\mathscr{R}(x, U_\lambda y)$ and of $|(x, U_\lambda y)|$, for x and y in $\mathfrak{M}_p$ and $\|x\| = \|y\| = 1$) are independent of p. Since any unit vector in $\mathfrak{H}$ can be approximated arbitrarily closely by a unit vector in $\mathfrak{M}_p$, for a sufficiently large negative p, it follows that the bounds of $(x, U_\lambda y)$ are the same in $\mathfrak{M}_1$ as in $\mathfrak{H}$. Further, the bounds are not altered, or are only permuted, by adding an integer to λ or by reversing the sign of λ. These statements remain true when x and y are constrained to be real or to be positive. It is therefore sufficient to determine the bounds of $(x, U_\lambda y)$ in $\mathfrak{H}$ with $0 < \lambda \leqslant \frac{1}{2}$. To this end, let x be a positive unit vector in $\mathfrak{M}_0$, and let y be the unit vector defined by $y_n = x_{-n}$. Then

$$(x, U_\lambda y) = \frac{\sin \pi\lambda}{\pi} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{0} \frac{x_m x_{-n}}{m-n+\lambda} = \frac{\sin \pi\lambda}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x_m x_n}{m+n+\lambda}.$$

Now Ingham† has shown that, when $0 < \lambda \leqslant \frac{1}{2}$, $\pi \operatorname{cosec} \pi\lambda$ is the exact upper bound of the quadratic form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{x_m x_n}{m+n+\lambda},$$

† *J. of London Math. Soc.* **11** (1936), 237–40.

for positive numbers x_n such that $\sum\limits_0^\infty x_n^2 = 1$. It follows from this and Schur's inequality that the upper bound of $(x, U_\lambda y)$ for positive unit vectors x and y in $\mathfrak{H}$ is 1. Thus the following facts have now been established:

When λ is real and not an integer, and $\sum\limits_1^\infty |x_n|^2 = \sum\limits_1^\infty |y_n|^2 = 1$, the upper bound of

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x_m y_n}{m-n+\lambda}\right|$$

is $\pi\,|\mathrm{cosec}\,\pi\lambda|$; this is not attained when all the x_n and all the y_n are non-negative, but in this case $\pi\,\mathrm{cosec}\,\pi\lambda$ is the upper bound of the form when it is positive and the lower bound when it is negative.

For the study of the quadratic form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{x_m x_n}{m-n+\lambda}$$

it is convenient to consider the unitary transformation V_λ defined by the matrix $(V_{\lambda;m,n})$, where, for all integers m and n,

$$V_{\lambda;m,n} = (-1)^{m+n}U_{\lambda;m,n}.$$

If x is any vector in $\mathfrak{H}$, and z is defined by $z_n = (-1)^n x_n$, then $\|x\| = \|z\|$ and $(x, U_\lambda x) = (z, V_\lambda z)$.

The mth row of $(V_{\lambda;m,n})$ consists of the numbers

$$\frac{\sin\pi(m-n+\lambda)}{\pi(m-n+\lambda)},$$

which are the Fourier constants of the function $e^{i(m+\lambda)t}$ with respect to the complete orthogonal system $\{e^{int}\}$ in the interval $-\pi < t < \pi$. Also, if x is any unit vector in $\mathfrak{H}$, the numbers x_n are the Fourier constants of a function $x(t)$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|x(t)|^2\,dt = 1.$$

Hence, by Parseval's theorem for scalar products,

$$\sum_{n=-\infty}^{\infty} V_{\lambda;m,n}\,x_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(m+\lambda)t}x(t)\,dt,$$

so that $\sum\limits_{n=-\infty}^{\infty} V_{\lambda;m,n}\,x_n$ is the mth Fourier constant of the function $e^{-i\lambda t}x(t)$.

(This provides an alternative proof that the transformations V_λ and U_λ are unitary.) It follows that

$$(V_\lambda x, x) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} V_{\lambda;m,n} x_n \overline{x_m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\lambda t} |x(t)|^2 \, dt,$$

whence $$\mathscr{R}(x, V_\lambda x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos \lambda t |x(t)|^2 \, dt.$$

This is bounded above by 1, and is bounded below by $\cos \pi\lambda$ when $|\lambda| < 1$ and by -1 otherwise; and it is easy to see that these bounds are not attained. They are, however, exact bounds. Indeed the upper bound of $(x, V_\lambda x)$ for positive unit vectors x is 1: for example, if $0 < \delta < \pi$ and x is defined by

$$x(t) = \begin{cases} \left(\dfrac{3\pi}{\delta}\right)^{\frac{1}{2}}\left(1 - \dfrac{|t|}{\delta}\right) & \text{when } |t| < \delta, \\ 0 & \text{when } \delta \leqslant |t| < \pi, \end{cases}$$

then $\|x\| = 1$, $x_n = \left(\dfrac{3}{\pi\delta^3}\right)^{\frac{1}{2}} \dfrac{1 - \cos n\delta}{n^2} \geqslant 0$, and $(x, V_\lambda x) \to 1$ as $\delta \to 0$.

Also, if z is defined by

$$z_n = (-1)^n x_n = (-1)^n \left(\frac{3}{\pi\delta^3}\right)^{\frac{1}{2}} \frac{1 - \cos n\delta}{n^2},$$

then $(x, U_\lambda x) = (z, V_\lambda z) \to \cos \pi\lambda$ as $\delta \to 0$.

As in the case of the bilinear form, the bounds of $(x, U_\lambda x)$ are the same in $\mathfrak{M}_1$ as in $\mathfrak{H}$. Hence:

When λ is real and not an integer, and the real numbers x_n are such that $\sum_1^\infty x_n^2 = 1$, the upper and lower bounds of

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{x_m x_n}{m - n + \lambda}$$

are $\pi \operatorname{cosec} \pi\lambda$ and $\pi \cot \pi\lambda$ when $|\lambda| < 1$, and are $\pm\pi|\operatorname{cosec} \pi\lambda|$ when $|\lambda| > 1$; when all the x_n are positive, $\pi \cot \pi\lambda$ is the upper bound of the form when $-1 < \lambda < 0$ and the lower bound when $0 < \lambda < 1$. The bounds are not attained.

THE MINIMUM OF A NON-HOMOGENEOUS BILINEAR FORM

By J. H. H. CHALK (London)

[Received 26 April 1951]

1. LET α, β, γ, δ, c_1, c_2 be real numbers with $\Delta = \alpha\delta-\beta\gamma \neq 0$ and consider the non-homogeneous bilinear form

$$B(x,y;z,t) = (\alpha x+\beta y+c_1)(\gamma z+\delta t+c_2), \tag{1}$$

where x, y, z, t are restricted to integer values satisfying

$$xt-yz = \pm 1. \tag{2}$$

It is convenient to define equivalent bilinear forms in the sense of Davenport and Heilbronn (**1**). Thus two bilinear forms will be called *equivalent* if one can be transformed into the other by one of the two following substitutions:

$$\left.\begin{array}{ll} x = px'+qy', & y = rx'+sy' \\ z = pz'+qt', & t = rz'+st' \end{array}\right\},$$

$$\left.\begin{array}{ll} x = pz'+qt', & y = rz'+st' \\ z = px'+qy', & t = rx'+sy' \end{array}\right\},$$

where p, q, r, s are integers satisfying $ps-qr = \pm 1$. We define $M(B)$ as the lower bound of the values of $|B(x,y;z,t)|$ for integers x, y, z, t satisfying (2). Clearly, equivalent bilinear forms B and B' say, have the property $M(B) = M(B')$. In this paper, our purpose is to establish the following theorem.

THEOREM. (i) *There exist integers x, y, z, t satisfying* (2) *such that*

$$|B(x,y;z,t)| \leqslant \tfrac{1}{4}|\Delta|. \tag{3}$$

If both the ratios α/β, γ/δ are irrational, there exist integers x, y, z, t satisfying (2) *and such that*

$$|B(x,y;z,t)| < \tfrac{1}{8}|\Delta|, \tag{4}$$

except when $(c_1,c_2) = (0,0)$.

(ii) *For all forms $B(x,y;z,t)$ with* $(c_1,c_2) = (0,0)$

$$M(B) \leqslant \frac{3-\sqrt{5}}{2\sqrt{5}}\,|\Delta|,$$

and equality occurs if and only if B is equivalent to a multiple of B_1, where

$$B_1 = \left(x+\frac{1+\sqrt{5}}{2}y\right)\left(z+\frac{1-\sqrt{5}}{2}t\right),$$

in which case $M(B_1)$ is attained.

If all forms equivalent to a multiple of B_1 are excluded, then

$$M(B) \leqslant (\tfrac{1}{2}-\tfrac{1}{4}\sqrt{2})|\Delta|,$$

and equality occurs if and only if B is equivalent to a multiple of B_2, where

$$B_2 = (x-\sqrt{2}y)(z+\sqrt{2}t),$$

in which case $M(B_2)$ is attained.

If all forms equivalent to a multiple of either B_1 or B_2 are excluded, then

$$M(B) \leqslant \frac{\sqrt{2}-1}{3}|\Delta|, \tag{5}$$

and equality occurs if and only if B is equivalent to a multiple of B_3, where

$$B_3 = (x-\sqrt{2}y)(z+(3-\sqrt{2})t),$$

in which case $M(B_3)$ is attained.

The results for the case $(c_1, c_2) = (0, 0)$ have been established by Davenport and Heilbronn (loc. cit.), and are included in the statement of the theorem for completeness. They also show that the chain of forms B_1, B_2, B_3 is complete, by proving, for any $\delta > 0$, that there exists a set of forms B with $(c_1, c_2) = (0, 0)$, no one of which is a multiple of another, for which

$$M(B) > \left(\frac{\sqrt{2}-1}{3}-\delta\right)|\Delta|,$$

and that this set has the cardinal number of the continuum.

It is curious that, in the case when both of the ratios α/β, γ/δ are irrational, the presence of arbitrary constants $(c_1, c_2) \neq (0, 0)$ permits a constant

$$\tfrac{1}{8}|\Delta| < \tfrac{1}{3}(\sqrt{2}-1)|\Delta| = \frac{1}{7{\cdot}24...}|\Delta|,$$

considerably smaller than that in (5). The proof of this result is a consequence of part (i) of the following lemma; part (ii) is relevant when one of α/β, γ/δ is rational, but this case is not considered. However, it is included here, for together with part (i) we can easily deduce a result for the critical determinant $\Delta(S)$ of the region S defined by

$$|(x+c)y| \leqslant 1 \quad (c \neq 0); \tag{6}$$

namely, we find that† $\Delta(S) = 2+\sqrt{2}$.

† See the Corollary to Lemma 5.

LEMMA 5. *Let* $L_1 = \alpha x+\beta y$, $L_2 = \gamma x+\delta y$ *be two real linear homogeneous forms in the variables* x, y, *where* $\Delta = \alpha\delta-\beta\gamma \neq 0$, *and let* $c \neq 0$ *be an arbitrary real constant.*

(i) *If* α/β *is irrational, there exist integers* (x,y) *such that*

$$(x,y) = 1, \tag{7}$$

and

$$|(L_1+c)L_2| < \tfrac{1}{4}|\Delta|. \tag{8}$$

(ii) *If* α/β *is rational, there exist integers* (x,y) *such that* (7) *holds, and*

$$|(L_1+c)L_2| \leqslant (1-\tfrac{1}{2}\sqrt{2})|\Delta|. \tag{9}$$

Equality in (9) *occurs if and only if*

$$L_1 = \lambda_1 y, \quad L_2 = \lambda_2(x+(\sqrt{2}-1)y), \quad c = -(1-\tfrac{1}{2}\sqrt{2})\lambda_1, \tag{10}$$
$$\lambda_1\lambda_2 = \Delta,$$

apart from an integral unimodular substitution on the variables x, y.

For the proof of this, we need four further lemmas.

2. LEMMA 1. *For all integers* x, y *with* $(x,y) \neq (0,0)$ *we have*

$$|((2+\sqrt{2})y-1)(x+(\sqrt{2}-1)y)| \geqslant 1.$$

Proof. Suppose, if possible, that there exists an integer pair $(x,y) \neq (0,0)$ such that

$$|((2+\sqrt{2})y-1)(x+(\sqrt{2}-1)y)| < 1.$$

If $y \leqslant 0$ and we replace x,y by $-x$, $-y$ we do not increase the absolute value of the product. So we may suppose that $y \geqslant 0$. Multiplying the inequality by $|(2-\sqrt{2})y-1|$ we obtain

$$|((2y-1)^2-2y^2)(x+(\sqrt{2}-1)y)| < |(2-\sqrt{2})y-1|,$$

so that

$$|x+(\sqrt{2}-1)y| < |(2-\sqrt{2})y-1|. \tag{11}$$

Multiplying the inequality by $|x-(\sqrt{2}+1)y|$, we obtain

$$|((2+\sqrt{2})y-1)((x-y)^2-2y^2)| < |x-(\sqrt{2}+1)y|,$$

so that, since x and y are not both zero,

$$|(2+\sqrt{2})y-1| < |x-(\sqrt{2}+1)y|. \tag{12}$$

Now, if $y = 0$ or 1, it is clear that (11) cannot be satisfied. Further, if $y > 1$, the conditions (11) and (12) give

$$|x| < |(2-\sqrt{2})y-1|+(\sqrt{2}-1)y = y-1,$$

and

$$|x| > |(2+\sqrt{2})y-1|-(\sqrt{2}+1)y = y-1.$$

This contradiction proves the lemma.

COROLLARY. *If L_1, L_2, c are as defined in* (10), *then*

$$|(L_1+c)L_2| \geqslant (1-\tfrac{1}{2}\sqrt{2})|\Delta|$$

for all integers x, y with $(x,y) \neq (0,0)$.

LEMMA 2 (**2**). *If A, B, C are three points of a two-dimensional lattice of determinant Δ, such that the triangle ABC contains no lattice point inside or on its boundary, other than A, B, C, then its area is $\tfrac{1}{2}\Delta$.*

Proof. Complete the parallelogram A, B, C, D, where CD is equal and parallel to AB. Since ABC is devoid of lattice points other than A, B, C, clearly $ABCD$ forms a fundamental cell of the lattice. Hence

$$\text{area } ABC = \tfrac{1}{2} \text{ area } ABCD = \tfrac{1}{2}\Delta.$$

3. LEMMA 3. *Let x_r, y_r be positive numbers satisfying*

$$x_r y_r \geqslant 1 \quad (r = 1, 2, 3, 4),$$

and let P_1, P_2, P_3, P_4 respectively denote the points in the (x,y)-plane with coordinates (x_1, y_1), $(-x_2, y_2)$, $(-x_3, -y_3)$, $(x_4, -y_4)$. Then the region bounded by the segments P_1P_2, P_2P_3, P_3P_4, P_4P_1 is a quadrilateral with area exceeding or equal to 4, with strict inequality unless

$$P_1 = (t, t^{-1}), \qquad P_2 = (-t, t^{-1}), \qquad P_3 = (-t, -t^{-1}), \qquad P_4 = (t, -t^{-1})$$

for any $t > 0$.

Proof. The area $\mathscr{A}(\mathscr{R})$ of the region $\mathscr{R}$, say, bounded by the segments is given by

$$\begin{aligned}\mathscr{A}(\mathscr{R}) &= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ -x_2 & y_2 & 1 \\ x_4 & -y_4 & 1 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} x_4 & -y_4 & 1 \\ -x_2 & y_2 & 1 \\ -x_3 & -y_3 & 1 \end{vmatrix} \\ &= \tfrac{1}{2}\{(x_2y_4 - x_4y_2) - (-x_1y_4 - x_4y_1) + (x_1y_2 + x_2y_1)\} + \\ &\qquad + \tfrac{1}{2}\{(x_2y_3 + x_3y_2) - (-x_4y_3 - x_3y_4) + (x_4y_2 - x_2y_4)\} \\ &= \tfrac{1}{2}\{x_1y_4 + x_4y_1 + x_1y_2 + x_2y_1\} + \tfrac{1}{2}\{x_2y_3 + x_3y_2 + x_4y_3 + x_3y_4\} \\ &\geqslant (x_1y_1x_4y_4)^{\frac{1}{2}} + (x_1y_1x_2y_2)^{\frac{1}{2}} + (x_2y_2x_3y_3)^{\frac{1}{2}} + (x_4y_4x_3y_3)^{\frac{1}{2}} \\ &\geqslant 4,\end{aligned}$$

by the hypotheses, and the inequality of the arithmetic and geometric means. Hence strict inequality holds unless

$$x_1y_4 = x_4y_1, \qquad x_1y_2 = x_2y_1, \qquad x_2y_3 = x_3y_2, \qquad x_4y_3 = x_3y_4,$$

and

$$x_r y_r = 1 \quad (r = 1, 2, 3, 4).$$

These imply that

$$x_1 = x_2 = x_3 = x_4 = t, \qquad y_1 = y_2 = y_3 = y_4 = t^{-1},$$

which establishes the lemma.

For the enunciation of the next lemma, it is convenient to define the term 'semi-infinite strip'. The lemma itself is stated here without proof as it is a well-known consequence of Kronecker's theorem.

DEFINITION. *If $a < b$ and α, β are not both zero, then the set of points (x, y) in two-dimensional space defined by*

$$a < \alpha x + \beta y + \gamma < b, \qquad \beta x - \alpha y > \delta$$

is termed a 'semi-infinite strip'.

LEMMA 4. *Let Λ denote a two-dimensional lattice (homogeneous or non-homogeneous), and $\mathscr{C}$ denote any semi-infinite strip of points in the plane of this lattice. Then, if $\mathscr{C}$ does not lie parallel to a one-dimensional sub-lattice of Λ, it contains an infinity of points of Λ.*

4. Proof of Lemma 5

(i) The points (X, Y) in 2-dimensional space arising from integer values of x, y, where

$$X = \alpha x + \beta y, \qquad Y = \gamma x + \delta y$$

form a plane lattice Λ of determinant $|\Delta|$. Thus, geometrically, the lemma asserts that, if there are no points of Λ, other than O, on the Y-axis, the region

$$|(X+c)Y| < \tfrac{1}{4}|\Delta| \tag{13}$$

contains a primitive point of Λ. The lemma is trivial if there are points of Λ on the X-axis (apart from O), so henceforward we shall exclude this case.

I shall prove that, if Λ^* is any lattice having no points on the Y-axis, other than O, and having no primitive points in the region

$$|(X-1)Y| < 1, \tag{14}$$

then $d(\Lambda^*) > 4$.

This suffices to prove (i), for suppose that there exists a lattice Λ of determinant $|\Delta|$ having no primitive points in (13) and having no points on the Y-axis, other than O, then the lattice Λ^* obtained from Λ by the affine transformation

$$X' = -\lambda X, \qquad Y' = \mu Y,$$

where $\quad d(\Lambda^*) = |\lambda\mu|\,|\Delta|, \qquad \lambda = c^{-1}, \qquad \mu = 4c/|\Delta|,$

has no primitive points in

$$|(X-1)Y| < \tfrac{1}{4}|\lambda\mu|\,|\Delta| = 1,$$

and no points on the Y-axis. Hence, we should have

$$4 < d(\Lambda^*) = |\lambda\mu|\,|\Delta| = |c^{-1}.4c/\Delta|\,|\Delta| = 4,$$

which is impossible.

Consider the region

$$R: \qquad |X-1|Y < 1, \quad X > 0, \quad Y > 0, \tag{15}$$

and the region

$$S: \qquad (X-1)Y < 1, \quad X > 0, \quad Y > 0, \tag{16}$$

where, trivially, $S \supset R$. Let $S-R$ denote the set of points which belongs to S but not to R. Then points of $S-R$ satisfy

$$(1-X)Y \geqslant 1, \qquad X > 0, \qquad Y > 0.$$

Hence $Y/X \geqslant [X(1-X)]^{-1} \geqslant 4$, so that they also satisfy

$$Y-4X \geqslant 0. \tag{17}$$

Now, if R contains a point of Λ^*, it is not primitive, since R is a subset of (14). All such points satisfy (17), for otherwise, if Q is a non-primitive point of Λ^* in R, the open segment OQ would be contained in R; but, for some integer n, the point $n^{-1}Q$ is a primitive point of Λ^* in R—a contradiction.

Hence all points of Λ^* in S satisfy (17). Now the semi-infinite strip given by $0 < X < 1$, $Y > 0$ is a subset of S. Since, by hypothesis, there are no points of Λ^* on the Y-axis, we know by Lemma 4 that this strip contains a point of Λ^*. Hence there is a point of Λ^* with co-ordinates (p, q), say, in S. Thus

$$(p-1)q < 1, \quad p > 0, \quad q > 0.$$

The subset of S satisfying

$$\frac{q}{p} \geqslant \frac{Y}{X} \geqslant 4$$

is a bounded region, by (16), and so contains at most a finite number of lattice points.

Let $\bar{\mu}$ denote the lower bound of the numbers μ, where μ is such that the line $Y-\mu X = 0$ contains a point of Λ^* in S. By the remark above, the lower bound is attained, and there exists a point of Λ^* in S satisfying $Y-\bar{\mu}X = 0$, where $\bar{\mu} \geqslant 4$. Obviously, we may suppose that this point is primitive. Then it is unique and we denote it by $P(a, b)$. By hypothesis, P does not belong to R and so it must belong to $S-R$, whence

$$a > 0, \qquad b \geqslant 1.$$

From the construction, we can now assert that the subset of S satisfying

$$Y-\bar{\mu}X \leqslant 0, \qquad Y \leqslant b$$

is devoid of lattice points other than P.

The argument is repeated with the regions R', S' obtained from R, S respectively, by reflection in the X-axis. In this way, we establish the existence of numbers $\bar{\mu}' \geqslant 4$, a', b' and a primitive lattice point $P'(a', b')$, where $a' > 0$, $b' \leqslant -1$ such that the subset of S' satisfying

$$Y+\bar{\mu}'X \geqslant 0, \qquad Y \geqslant b'$$

is devoid of lattice points other than P'. By hypothesis, P' does not belong to R' and so it must belong to $S'-R'$.

Consider the region defined by

$$Y-\bar{\mu}X \leqslant 0, \qquad Y+\bar{\mu}'X \geqslant 0, \qquad b' \leqslant Y \leqslant b.$$

Apart from O, P, P', points of Λ^* in this region do not belong to S or S'. Hence they satisfy

$$(X-1)|Y| \geqslant 1. \tag{18}$$

Let Q and Q' denote any two such points, with Y-coordinates of opposite sign. These exist since (i) we have excluded the case when there are points of Λ^*, other than O, on the X-axis, (ii) the regions

$$\left.\begin{matrix} Y-\bar{\mu}X \leqslant 0 \\ 0 < Y \leqslant b \end{matrix}\right\}, \qquad \left.\begin{matrix} Y+\bar{\mu}'X \geqslant 0 \\ -b' \leqslant Y < 0 \end{matrix}\right\}$$

each contain a semi-infinite strip parallel to the X-axis, and (iii) we may apply Lemma 4.

Consider now the closed region Π, say, bounded by the segments OP, PQ, QQ', $Q'P'$, $P'O$. It contains at most a finite number of lattice points, all of which satisfy (18). If necessary, the points Q, Q' are now replaced, in a finite number of steps, by pairs of lattice points in Π, until we arrive at a situation where Π has the following properties:

(i) it contains no lattice point except O, P, Q, Q', P';

(ii) P, Q' belong to the upper half-plane, and P', Q' to the lower half-plane.

Consider the quadrilateral $PP'QQ'$. It contains no point of Λ^* other than P, P', Q, Q', since $\angle POP' < \pi$, and it is situated (by construction) so that no two of the vertices have coordinates with the same sign for $(1-X)$ *and* for Y.

Let A denote the point with coordinates $(-1, 0)$. Then the points $P+A$, $P'+A$, $Q+A$, $Q'+A$ satisfy the hypotheses for P_1, P_2, P_3, P_4 in Lemma 3. Hence

$$\text{area } PP'QQ' \geqslant 4,$$

with equality if and only if the four points P, P', Q, Q' are given by

$$(1-t, t^{-1}),\quad (1-t, -t^{-1}),\quad (1+t, t^{-1}),\quad (1+t, -t^{-1}).$$

In this case the lattice point $P-P'$ with coordinates $(0, 2t^{-1}) \neq (0,0)$ belongs to the Y-axis, contrary to hypothesis, and so we have

$$\text{area } PP'QQ' > 4.$$

Since, also, the quadrilateral $PP'QQ'$ contains no point of Λ^*, other than the vertices, we have, by Lemma 2,

$$d(\Lambda^*) = \text{area } PP'QQ'.$$

Thus $d(\Lambda^*) > 4$; which establishes part (i) of the theorem.

(ii) It is required to prove that, if the Y-axis contains a point of a plane lattice Λ, other than O, then there exists a primitive point of Λ in the region

$$|(X+c)Y| \leqslant (1-\tfrac{1}{2}\sqrt{2})|\Delta|;$$

the point being in its interior, unless Λ is the special lattice defined by

$$X = \lambda_1 y, \qquad Y = \lambda_2\{x+(\sqrt{2}-1)y\}, \qquad c = -(1-\tfrac{1}{2}\sqrt{2})\lambda_1.$$

This inequality is trivially satisfied if there exist points of Λ, other than O, on the X-axis, so this case will be excluded henceforward.

I shall prove that, if Λ^* is any lattice (with points on the Y-axis) having *no* primitive points in the region

$$|(X-1)Y| \leqslant 1,$$

then $d(\Lambda^*) \geqslant 2+\sqrt{2}$ with strict inequality unless Λ^* is the lattice of points (X, Y), defined by

$$X = (2+\sqrt{2})y, \qquad Y = x+(\sqrt{2}-1)y \quad (x, y = 0, \pm 1, \pm 2, \ldots).$$

A similar argument to that in part (i) shows that these two statements are equivalent. It is convenient to consider two cases.

Case 1. Suppose that S or S' contains a point of Λ^*. The following argument applies to both cases and so it suffices to suppose that S contains a point P of Λ^*. It may be verified that the point tP, for $0 < t < 1$, also belongs to S. If P denotes a point of Λ^* in S, then $n^{-1}P$ for some integer $n \geqslant 1$ is a primitive point of Λ^* in S. Since R is by hypothesis devoid of primitive points, such a point must belong to $S-R$. Now any line parallel to the Y-axis with a point in $S-R$ has a segment of infinite length in $S-R$. Suppose that this line, l say, contains a point of Λ^*. Then, since the Y-axis contains a point of Λ^* other than O, the line l contains an infinite set of equidistant lattice points. Hence $S'-R'$ also contains a point of Λ^*. As in part (i), this

fact permits us to define the numbers $\bar{\mu}$, $\bar{\mu}'$, and so we can repeat the argument given there. Hence

$$d(\Lambda^*) \geqslant 4 > 2+\sqrt{2}.$$

Case 2. Suppose that neither of S, S' contains a point of Λ^*. Then there exists a point of Λ^*, other than O, on the Y-axis, and all lattice points other than those on the Y-axis have X-coordinates $\geqslant 1$. Hence we may write the lattice Λ^* in the form

$$X = py, \qquad Y = ax+qy \quad (x, y = 0, \pm 1, \pm 2, \ldots),$$

where a, p, q are positive numbers with

$$a \geqslant 1, \qquad p \geqslant 1.$$

By hypothesis, if $(x, y) = 1$, then

$$|(py-1)(ax+qy)| \geqslant 1.$$

By replacing x by $\pm x+ny$, where n is a suitable integer, we may suppose that

$$0 \leqslant q \leqslant \tfrac{1}{2}a \tag{19}$$

without affecting the condition $(x, y) = 1$. To obtain the result it suffices to use the weaker hypotheses,

$$(p-1)q \geqslant 1, \qquad (2p-1)(a-2q) \geqslant 1.$$

From these we obtain

$$d(\Lambda^*) = ap \geqslant a \max\left\{1+\frac{1}{q},\ \frac{1}{2}\left(\frac{1}{a-2q}+1\right)\right\},$$

if q satisfies (19). Now, $1+q^{-1}$ is a decreasing function of q, and

$$\frac{1}{2}\left(\frac{1}{a-2q}+1\right)$$

is an increasing function of q in this range. Hence

$$\max\left\{1+\frac{1}{q},\ \frac{1}{2}\left(\frac{1}{a-2q}+1\right)\right\} \geqslant 1+\frac{1}{q'},$$

where q' is a solution of the equation

$$1+\frac{1}{q'} = \frac{1}{2}\left(\frac{1}{a-2q'}+1\right).$$

This has two solutions

$$q' = \tfrac{1}{4}\{\pm\sqrt{[(5-a)^2+16a]}-(5-a)\}.$$

Since $q' > 0$, the negative sign is inadmissible, and so

$$1+\frac{1}{q'} = \frac{1}{4a}\{\sqrt{(a^2+6a+25)}+5+3a\}.$$

Hence $\quad d(\Lambda^*) = ap \geqslant \frac{1}{4}\{\sqrt{(a^2+6a+25)}+5+3a\} \geqslant 2+\sqrt{2},$

and $d(\Lambda^*) > 2+\sqrt{2}$, unless

$$a = 1, \qquad q = q' = \sqrt{2}-1, \qquad p = 2+\sqrt{2}.$$

This completes the proof of the lemma.

COROLLARY. *If S denotes the region defined by* (6), *then*

$$\Delta(S) = 2+\sqrt{2}.$$

By (8) and (9), a lattice Λ admissible for S satisfies

$$d(\Lambda) \geqslant 2+\sqrt{2},$$

so that $\Delta(S) \geqslant 2+\sqrt{2}$. The linear forms L_1 and L_2 in (10) define a lattice of determinant $2+\sqrt{2}$ which, by Lemma 1, is admissible for S. Hence $\Delta(S) \leqslant 2+\sqrt{2}$, which gives the required result.

5. Proof of the theorem

It suffices to consider the case $(c_1, c_2) \neq (0,0)$. For, if $(c_1, c_2) = (0,0)$, the results given by Davenport and Heilbronn clearly establish the assertions of the theorem.

Suppose that at least one of the ratios α/β, γ/δ is rational. Without loss of generality, we suppose that this ratio is α/β. By introducing a suitable factor $\lambda \neq 0$ into the form $\alpha x+\beta y+c_1$ and λ^{-1} into the form $\gamma z+\delta t+c_2$, we may suppose that α, β are integers with $(\alpha, \beta) = 1$. We now put $x' = \alpha x+\beta y$, $z' = \alpha z+\beta t$ and choose integers p, q to satisfy $\alpha q-\beta p = \pm 1$. The transformation is completed by putting

$$y' = px+qy, \qquad t' = pz+qt.$$

In this way the form B is transformed into an equivalent one of the shape

$$(x'+c_1')(\gamma' z'+\delta' t'+c_2'),$$

where x', y', z', t' satisfy $x't'-y'z' = \pm 1$. We now choose an integer x' so that $|x'+c_1'| \leqslant \frac{1}{2}$, put $z' = 1$, and choose t' so that

$$|\gamma'+\delta' t'+c_2'| \leqslant \tfrac{1}{2}|\delta'|.$$

Our choice of y' is then determined by the equation $y' = x't' \mp 1$. Hence we can satisfy

$$|(x'+c_1')(\gamma' z'+\delta' t'+c_2')| \leqslant \tfrac{1}{4}|\delta'| = \tfrac{1}{4}|\Delta|,$$

which establishes the inequality (3).

Suppose now that both of the ratios α/β, γ/δ are irrational. If necessary, we interchange the symbols α, β, c_1, x, y with γ, δ, c_2, z, t respectively, to ensure that

$$c_1 \neq 0.$$

By Lemma 5 (i), there exists an integer pair (x_0, y_0) with $(x_0, y_0) = 1$ satisfying

$$|(\alpha x_0+\beta y_0+c_1)(\gamma x_0+\delta y_0)| < \tfrac{1}{4}|\Delta|.$$

For each such (x_0, y_0) we can find integers (t_0, z_0) such that

$$x_0 t_0 - y_0 z_0 = 1 \quad (\text{or } -1).$$

We can choose an integer m so that

$$|m(\gamma x_0+\delta y_0)+\gamma z_0+\delta t_0+c_2| \leqslant \tfrac{1}{2}|\gamma x_0+\delta y_0|,$$

since $\gamma x_0+\delta y_0 \neq 0$, and put

$$x = x_0, \qquad y = y_0, \qquad z = z_0+mx_0, \qquad t = t_0+my_0. \tag{20}$$

We note that

$$\begin{aligned} xt-yz &= x_0(t_0+my_0)-y_0(z_0+mx_0) \\ &= x_0 t_0-y_0 z_0 = 1 \quad (\text{or } -1). \end{aligned}$$

Denote $|(\alpha x+\beta y+c_1)(\gamma z+\delta t+c_2)|$ by P. Then for the values of x, y, z, t given by (20) we have

$$\begin{aligned} P &= |(\alpha x_0+\beta y_0+c_1)(\gamma(z_0+mx_0)+\delta(t_0+my_0)+c_2)| \\ &= |(\alpha x_0+\beta y_0+c_1)(m(\gamma x_0+\delta y_0)+\gamma z_0+\delta t_0+c_2)| \\ &\leqslant \tfrac{1}{2}|(\alpha x_0+\beta y_0+c_1)(\gamma x_0+\delta y_0)| \\ &< \tfrac{1}{8}|\Delta|. \end{aligned}$$

This proves the theorem.

REFERENCES

1. H. Davenport and H. Heilbronn, *Quart. J. of Math.* (Oxford), 18 (1947), 107–21.
2. Lemmas 2 and 3 are contained, substantially, in a paper by D. B. Sawyer, *J. of London Math. Soc.* 23 (1948), 250–1.

NOTE ON A RESULT OF CHALK

By H. DAVENPORT (*London*)

[Received 2 May 1951]

1. LET X, Y be linear forms in u, v of determinant 1, with real coefficients, and suppose that neither form represents zero for integers u, v other than 0, 0. Let c be a real number other than 0. Dr. Chalk† has recently proved the curious result that *there exist integers* u, v *such that*

$$(u,v) = 1, \qquad |(X+c)Y| < \tfrac{1}{4}. \tag{1}$$

The interest of the result lies mainly in the condition of relative primality imposed on u and v. Without this condition, the solubility of the inequality $|(X+c)Y| < \frac{1}{4}$ in integers u, v other than 0, 0 is a consequence of Minkowski's theorem on the product of two non-homogeneous linear forms. Indeed, this inequality has infinitely many solutions with $|X+c|$ arbitrarily small, provided that $X+c$ does not represent zero.

The first question that suggests itself is whether the constant $\frac{1}{4}$ in (1) is the best possible. We prove in Theorem 1 that this is the case. The idea on which the proof is based is a simple and natural one. We take Y to be a homogeneous linear form which approximates badly to zero, such as $Y = u-\phi v$, where $\phi = \frac{1}{2}(\sqrt{5}-1)$. We take $X+c$ to be a non-homogeneous linear form which approximates badly to zero, such as the form of Lemma 1 below. In this way we construct forms $X+c$ and Y for which the inequality $|(X+c)Y| < \frac{1}{4}-\delta$, where δ is any fixed positive number, can have at most a finite number of solutions in integers u, v. On making a particular choice of the form $X+c$, we are able to prove that the inequality has only two solutions other than 0, 0; and these two solutions do not satisfy the condition $(u,v) = 1$.

It may be of some interest to investigate the analogous problem which arises when the condition $(u,v) = 1$ in (1) is omitted, or rather, is replaced by the obvious condition that u and v shall not both be 0. We prove in Theorem 2 that the inequality

$$|(X+c)Y| < (4{\cdot}1)^{-1} \tag{2}$$

is always soluble in integers u, v other than 0, 0. Here, of course, the constant is not the best possible. We prove in § 4 that the inequality (2) is not always soluble if the number $(4{\cdot}1)^{-1}$ on the right is replaced by $(5{\cdot}06)^{-1}$. It may be remarked, however, that the constant $\frac{1}{4}$ still has a

† See above, pp. 119–29.

certain significance for this problem. As we have already observed, it is the least constant for which the inequality has infinitely many solutions; and indeed it follows from Theorem 1 that it is the least constant for which the inequality has *three or more* solutions with u, v not both zero.

2. We need the following

LEMMA 1.† *Let the linear form X and the constant c be defined by*

$$X+c = u+\theta v-\tfrac{1}{2}(1+\theta),$$

where
$$\theta = 2k+\sqrt{(4k^2+1)}, \tag{3}$$

k being a positive integer. Let X' and c' be similarly defined in terms of θ', where $\theta' = 2k-\sqrt{(4k^2+1)}$. Then

$$|(X+c)(X'+c')| \geqslant k \tag{4}$$

for all integers u, v.

The example by which we prove that the constant $\frac{1}{4}$ in (1) is the best possible is the following. Define the linear form X and the constant c by

$$X+c = u+\theta v-\tfrac{1}{2}(1+\theta)+k. \tag{5}$$

This differs from the definition in the lemma in that the integer k has been added; but plainly (4) is still valid, since u can be changed into $u+k$. Define the linear form Y by

$$Y = u-\phi v \quad \text{where } \phi = \tfrac{1}{2}(\sqrt{5}-1) = 0{\cdot}618\ldots. \tag{6}$$

The determinant of the linear forms X, Y is now no longer 1 but $\theta+\phi$, and this factor must therefore be introduced into the right-hand side of any inequality of the kind under consideration, if it is to be appropriate to these particular linear forms. We prove

THEOREM 1. *Let the linear forms X, Y and the constant c be defined by* (5) *and* (6). *Let δ be any positive number. Then, if k is sufficiently large, the only solutions in integers u, v of the inequality*

$$|(X+c)Y| < (\tfrac{1}{4}-\delta)(\theta+\phi) \tag{7}$$

are the three solutions given by

$$u = v = 0; \qquad u = k, \quad v = 0; \qquad u = k+1, \quad v = 0.$$

Proof. We suppose the inequality (7) to hold. We have

$$|(X+c)-Y| = |(\theta+\phi)v-\tfrac{1}{2}(1+\theta)+k|.$$

† See H. Davenport, *Proc. K. Nederlandsche Akad. van Wet.* **49** (1946), 815–21, Lemma 3 (with k replaced by $2k$).

By (3), we have $\theta \sim 4k$ when k is large. Hence $\frac{1}{2}(1+\theta)-k \sim \frac{1}{4}(\theta+\phi)$, and it follows that the last expression above is least when $v = 0$. Since this least value is greater than $\frac{1}{4}(\theta+\phi)$, we have

$$|X+c|+|Y| > \tfrac{1}{4}(\theta+\phi). \tag{8}$$

It follows from (7) and (8), since $\theta+\phi$ is large, that

$$\text{either } |X+c| < 1 \quad \text{or } |Y| < 1.$$

We now consider several cases.

Case 1. Suppose that $|X+c| < 1$ *and* $0 < |v| < \sqrt{k}$. We have

$$\begin{aligned} X+c &\equiv \theta v - \tfrac{1}{2}(1+\theta) \pmod 1 \\ &\equiv -\theta' v - \tfrac{1}{2} + \tfrac{1}{2}\theta' \pmod 1, \end{aligned}$$

since $\theta+\theta' = 4k$. Hence

$$|X+c| \geqslant \tfrac{1}{2} - |\theta'|(|v|+\tfrac{1}{2}) > \tfrac{1}{2} - \delta \tag{9}$$

for large k, since $|v| < \sqrt{k}$ and $|\theta'| \sim (4k)^{-1}$.

Also, since $|X+c| < 1$, we can write

$$u = -\theta v - k + \tfrac{1}{2}(1+\theta) + \xi, \quad \text{where } |\xi| < 1. \tag{10}$$

Substituting in Y, we obtain

$$Y = -\theta v - k + \tfrac{1}{2}(1+\theta) + \xi - \phi v,$$

$$|Y| > (\theta+\phi)|v| - \{\tfrac{1}{2}(1+\theta) - k + 1\}.$$

The last expression in brackets is asymptotic to $\frac{1}{4}(\theta+\phi)$ for large k. Hence, since $v \neq 0$,

$$|Y| > (\tfrac{3}{4}-\delta)(\theta+\phi). \tag{11}$$

The inequalities (9) and (11) contradict (7). There is, indeed, something to spare in this case.

Case 2. Suppose that $|X+c| < 1$ *and* $|v| \geqslant \sqrt{k}$. By Lemma 1, we have

$$|(X+c)Y| \geqslant k\left|\frac{Y}{X'+c'}\right| = k\left|\frac{u-\phi v}{u+\theta' v - \frac{1}{2}(1+\theta')+k}\right|.$$

Since (10) is still valid, this is

$$k\left|\frac{-\theta v - k + \frac{1}{2}(1+\theta) + \xi - \phi v}{-\theta v - k + \frac{1}{2}(1+\theta) + \xi + \theta' v - \frac{1}{2}(1+\theta') + k}\right|$$

$$= k\left|\frac{(\theta+\phi)v + k - \frac{1}{2}(1+\theta) - \xi}{(\theta-\theta')v - \frac{1}{2}(\theta-\theta') - \xi}\right| > k\frac{(\theta+\phi)|v| - (\frac{1}{2}\theta - k + \frac{3}{2})}{(\theta-\theta')|v| + \frac{1}{2}(\theta-\theta') + 1} > k\frac{|v|-A}{|v|+B},$$

where A and B are bounded as $k \to \infty$. Since $|v| \geqslant \sqrt{k}$, and $k \sim \frac{1}{4}(\theta+\phi)$, we have a contradiction to (7).

Case 3. Suppose that $|Y| < 1$ *and* $|v| \geqslant 2$. Write

$$u = \phi v+\eta, \quad \text{where } |\eta| < 1.$$

Then $$|(X+c)Y| \geqslant \left|\frac{X+c}{Y'}\right| = \left|\frac{\phi v+\eta+\theta v-\frac{1}{2}(1+\theta)+k}{\phi v+\eta-\phi' v}\right|,$$

where $\phi' = -\frac{1}{2}(\sqrt{5}+1) = -1{\cdot}618\ldots$. Hence

$$|(X+c)Y| \geqslant \frac{(\theta+\phi)|v|-(\frac{1}{2}\theta-k+\frac{3}{2})}{\sqrt{5}|v|+1}.$$

Since $\frac{1}{2}\theta-k+\frac{3}{2} \sim \frac{1}{4}(\theta+\phi)$ as $k \to \infty$, it suffices to prove that

$$\frac{|v|-\frac{1}{4}}{\sqrt{5}|v|+1} > \frac{1}{4},$$

since then we have a contradiction to (7). In fact the least value of the left-hand side occurs when $|v| = 2$, and is $0{\cdot}31\ldots$.

Case 4. Suppose that $|Y| < 1$ *and* $v = \pm 1$. Since $Y = u-\phi v$, where $\phi = 0{\cdot}618\ldots$, the only possibilities are $u = 0$ or 1 when $v = 1$ and $u = 0$ or -1 when $v = -1$. The corresponding values of $|(X+c)Y|$ are

$$(\tfrac{1}{2}\theta+k-\tfrac{1}{2})\phi \sim \tfrac{3}{4}(0{\cdot}618\ldots)(\theta+\phi),$$
$$(\tfrac{1}{2}\theta+k+\tfrac{1}{2})(1-\phi) \sim \tfrac{3}{4}(0{\cdot}382\ldots)(\theta+\phi),$$
$$(\tfrac{3}{2}\theta-k+\tfrac{1}{2})\phi \sim \tfrac{5}{4}(0{\cdot}618\ldots)(\theta+\phi),$$
$$(\tfrac{3}{2}\theta-k+\tfrac{3}{2})(1-\phi) \sim \tfrac{5}{4}(0{\cdot}382\ldots)(\theta+\phi).$$

Since each of the constant factors on the right is greater than $\frac{1}{4}$, we have a contradiction to (7).

Case 5. Suppose that $v = 0$. Here

$$|(X+c)Y| = |u(u-\tfrac{1}{2}(1+\theta)+k)|.$$

The least values of this occur for $u = 0$, k, and $k+1$, and these were excluded in the enunciation. The next least values are asymptotically $\frac{1}{4}(\theta+\phi)$ when $u = 1$ or -1, and $\frac{3}{8}(\theta+\phi)$ when $u = k-1$ or $k+2$. Hence we again obtain a contradiction to (7).

The cases considered exhaust all the possibilities, and the proof of Theorem 1 is complete.

3. We return to the general case, when X and Y are any two linear forms of determinant 1, neither of which represents zero. The symbols θ and ϕ will no longer be restricted to the special meanings given to them in § 2.

The essential lemma for the proof of Theorem 2 is as follows.

LEMMA 2. *There exists an integral unimodular transformation from u, v to new variables u', v', which transforms $(X+c)Y$ into*

$$\pm\frac{(u+\theta v-\alpha)(u-\phi v)}{\theta+\phi}, \tag{12}$$

where
$$\theta > 1, \qquad 0 < \phi < 1, \qquad 1 \leqslant \alpha < \theta. \tag{13}$$

Proof. We consider first the homogeneous linear forms X, Y. We apply an integral unimodular substitution which transforms the indefinite quadratic form XY into one which is 'reduced' in the sense of Gauss. Then in the new variables, say u_0, v_0, we have

$$X = \lambda_0(u_0+\theta_0 v_0), \qquad Y = \mu_0(u_0-\phi_0 v_0), \tag{14}$$

where $\theta_0 > 1$ and $0 < \phi_0 < 1$. The numbers θ_0 and ϕ_0 are irrational, since X and Y do not represent zero.

Let the continued fraction developments of θ_0 and ϕ_0 be

$$\theta_0 = a_0+\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots,$$

$$\phi_0 = \frac{1}{a_{-1}+}\,\frac{1}{a_{-2}+}\cdots.$$

Let θ_n and ϕ_n be defined for every integer n (positive, negative, or zero) by

$$\theta_n = a_n+\frac{1}{a_{n+1}+}\cdots,$$

$$\phi_n = \frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots.$$

Then, from the classical theory, there exists for every n a unimodular substitution from u_0, v_0 to new variables u_n, v_n which transforms the linear forms (14) into

$$X = \lambda_n(u_n+\theta_n v_n), \qquad Y = \mu_n(u_n-\phi_n v_n). \tag{15}$$

In fact, we have

$$u_n+\theta_n v_n = u_n+\left(a_n+\frac{1}{\theta_{n+1}}\right)v_n = \frac{1}{\theta_{n+1}}(u_{n+1}+\theta_{n+1}v_{n+1}),$$

$$u_n-\phi_n v_n = u_n+\left(a_n-\frac{1}{\phi_{n+1}}\right)v_n = -\frac{1}{\phi_{n+1}}(u_{n+1}-\phi_{n+1}v_{n+1}),$$

where the variables u_n, v_n and the variables u_{n+1}, v_{n+1} are connected by the unimodular substitution

$$u_{n+1} = v_n, \qquad v_{n+1} = u_n+a_n v_n.$$

The relation connecting successive values of the factors λ_n, μ_n in (15) is obviously

$$\lambda_n = \theta_{n+1}\lambda_{n+1}, \qquad \mu_n = -\phi_{n+1}\mu_{n+1}. \tag{16}$$

Since the determinant of the linear forms in (15) is 1, we have

$$\lambda_n \mu_n = \pm\frac{1}{\theta_n+\phi_n}. \tag{17}$$

By (15),
$$X+c = \lambda_n(u_n+\theta_n v_n-\alpha_n), \tag{18}$$
where α_n is defined for every n by $c = -\lambda_n \alpha_n$. By (16),

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{\lambda_n}{\lambda_{n+1}} = \theta_{n+1}. \tag{19}$$

Since $\theta_{n+1} > 1$, it follows that $|\alpha_n|$ increases with n. Also, by a well-known result in the theory of continued fractions, $\theta_n\,\theta_{n+1} > 2$ for every n. Hence $|\alpha_n| \to \infty$ as $n \to +\infty$ and $|\alpha_n| \to 0$ as $n \to -\infty$. It follows that there exists a value of n for which

$$|\alpha_{n-1}| < 1 \leqslant |\alpha_n|.$$

By (19), this implies $1 \leqslant |\alpha_n| < \theta_n$.

With the particular value of n just defined, the expression for $(X+c)Y$ in terms of the variables u_n and v_n has all the properties stated in the enunciation, except that α is not known to be positive. This, however, can be ensured by using the obvious substitution which changes the signs of both variables.

LEMMA 3. *Let a and b be any real numbers. Then there exists an integer u such that*

$$|(u-a)(u-b)| \leqslant \begin{cases} \frac{1}{2} & \text{if } (a-b)^2 \leqslant 2,\\ \frac{1}{2}\sqrt{\{(a-b)^2-1\}} & \text{if } (a-b)^2 \geqslant 2.\end{cases}$$

Proof. To obtain this well-known result, it suffices to choose the integer u so that $u' = u-\frac{1}{2}(a+b)$ lies in the interval

$$|u'| \leqslant \tfrac{1}{2} \quad \text{if } (a-b)^2 \leqslant 2,$$

$$\tfrac{1}{2}\{((a-b)^2-1)^{\frac{1}{2}}-1\} \leqslant u' \leqslant \tfrac{1}{2}\{((a-b)^2-1)^{\frac{1}{2}}+1\} \quad \text{if } (a-b)^2 \geqslant 2.$$

THEOREM 2. *Let X, Y be linear forms in u, v of determinant* 1, *neither of which represents zero for integers u, v not both zero; and let c be any real number other than* 0. *Then there exist integers u, v, not both zero, such that*

$$|(X+c)Y| < (4\cdot 1)^{-1}.$$

Proof. By Lemma 2, it suffices to prove that, if θ, ϕ, α satisfy (13), then there exist integers u, v, not both zero, such that

$$|(u+\theta v-\alpha)(u-\phi v)| < \frac{1}{4\cdot 1}(\theta+\phi).$$

We assume the contrary, namely that

$$|(u+\theta v-\alpha)(u-\phi v)| \geqslant \frac{1}{4\cdot 1}(\theta+\phi) \tag{20}$$

for all integers u, v, not both zero and deduce a contradiction.

Case 1. Suppose that $\alpha \geqslant \sqrt{2}$ and $\theta+\phi-\alpha \geqslant \sqrt{2}$. Taking first $v = 0$, we consider how small we can make the product

$$|u(u-\alpha)|$$

for an integer u different from zero. The latter condition precludes us from appealing directly to Lemma 3, which would give $\frac{1}{2}\sqrt{(\alpha^2-1)}$. However, this result is still valid, since the interval

$$\tfrac{1}{2}\{\alpha+\sqrt{(\alpha^2-1)}-1\} \leqslant u \leqslant \tfrac{1}{2}\{\alpha+\sqrt{(\alpha^2-1)}+1\},$$

occurring in the proof of Lemma 3, does not contain the origin. Hence, by (20),

$$\sqrt{(\alpha^2-1)} \geqslant \frac{1}{2\cdot 05}(\theta+\phi). \tag{21}$$

Similarly, taking $v = 1$ and appealing to Lemma 3, we get

$$\sqrt{\{(\theta+\phi-\alpha)^2-1\}} \geqslant \frac{1}{2\cdot 05}(\theta+\phi). \tag{22}$$

The inequalities (21) and (22) obviously imply that $\theta+\phi$ is fairly large, and that α is about $\frac{1}{2}(\theta+\phi)$. To see this explicitly, we use the inequality that

$$(\alpha^2-1)((A-\alpha)^2-1) \leqslant (\tfrac{1}{4}A^2-1)^2$$

for $1 < \alpha < A-1$. This gives

$$\tfrac{1}{4}(\theta+\phi)^2-1 \geqslant \frac{1}{(2\cdot 05)^2}(\theta+\phi)^2,$$

$$(\theta+\phi)^2 \geqslant \left\{\frac{1}{4}-\frac{1}{(2\cdot 05)^2}\right\}^{-1} > 82,$$

$$\theta+\phi > 9. \tag{23}$$

Also, by (21),

$$\alpha > \frac{1}{2\cdot 05}(\theta+\phi). \tag{24}$$

We now use three other cases of (20), namely

$$(1+\theta-\alpha)(1-\phi) \geqslant \frac{1}{4\cdot 1}(\theta+\phi), \tag{25}$$

$$(\theta-\alpha)\phi \geqslant \frac{1}{4\cdot 1}(\theta+\phi), \tag{26}$$

$$(1+2\theta-\alpha)|1-2\phi| \geqslant \frac{1}{4\cdot 1}(\theta+\phi). \tag{27}$$

Suppose first that $\phi < \frac{1}{2}$. Combining (26) and (27) and using (24), we get

$$1 > \frac{2(\theta+\phi)}{2\cdot 1\theta-2\phi}+\frac{\theta+\phi}{6\cdot 2\theta-2\phi+4\cdot 1}.$$

Since the right-hand side increases with ϕ, this implies

$$1 > \frac{2}{2\cdot 1}+\frac{\theta}{6\cdot 2\theta+4\cdot 1}.$$

By (23) we have $\theta > 8\cdot 5$; hence the right-hand side is greater than

$$\frac{2}{2\cdot 1}+\frac{8\cdot 5}{(6\cdot 2)(8\cdot 5)+(4\cdot 1)} > 0\cdot 95+0\cdot 14,$$

and we have a contradiction.

Suppose next that $\phi > \frac{1}{2}$. Combining (25) and (27), and using (24), we get

$$1 > \frac{2(\theta+\phi)}{2\cdot 1\theta-2\phi+4\cdot 1}+\frac{\theta+\phi}{6\cdot 2\theta-2\phi+4\cdot 1}.$$

Again the right-hand side increases with ϕ, and so

$$1 > \frac{2\theta+1}{2\cdot 1\theta+3\cdot 1}+\frac{\theta+\frac{1}{2}}{6\cdot 2\theta+3\cdot 1}.$$

Now $\theta > 8$ by (23), and therefore the right-hand side is greater than

$$\frac{17}{19\cdot 9}+\frac{1}{6\cdot 2} > 0\cdot 85+0\cdot 16,$$

giving again a contradiction.

Case 2. Suppose that $\alpha < \sqrt{2}$ and $\theta+\phi-\alpha \geqslant \sqrt{2}$. Taking $u = 1$, $v = 0$ in (20), we obtain

$$\alpha-1 \geqslant \frac{1}{4\cdot 1}(\theta+\phi) \geqslant \frac{1}{4\cdot 1}(\alpha+\sqrt{2}),$$

whence

$$\alpha \geqslant \frac{4\cdot 1+\sqrt{2}}{4\cdot 1-1} > \sqrt{2},$$

a contradiction.

Case 3. Suppose that $\theta+\phi-\alpha < \sqrt{2}$. As in (26), we have

$$(\theta-\alpha)\phi \geqslant \frac{1}{4\cdot 1}(\theta+\phi).$$

Put $S = \theta+\phi-\alpha$. The left-hand side above is at most $\frac{1}{4}S^2$, and the right-hand side is

$$\frac{1}{4\cdot 1}(S+\alpha) \geqslant \frac{1}{4\cdot 1}(S+1).$$

Hence $$\tfrac{1}{4}S^2-\frac{1}{4\cdot 1}S-\frac{1}{4\cdot 1}\geqslant 0.$$

This contradicts the hypothesis that $S<\sqrt{2}$.

The proof of Theorem 2 is now complete.

4. No doubt the constant $(4\cdot 1)^{-1}$ in Theorem 2 could be improved by more delicate or more elaborate arguments, but it does not seem to be easy to find the best possible constant for this problem. As regards results in the opposite direction, the example of Theorem 1 shows that the result could not hold with a constant less than $\frac{1}{8}$. For the cases that are relevant are the excluded cases of Theorem 1, namely $u=k$ or $k+1$ and $v=0$. In those cases, $|X+c|$ is about $\frac{1}{2}$ and $|Y|$ is about $\frac{1}{4}(\theta+\phi)$.

A slightly different example gives a better result. Define Y as before, but define X and c by
$$X+c=u+\theta v-\tfrac{1}{2}(1+\theta),$$
where $\theta=5+\sqrt{26}=10\cdot 0990\ldots$. On using the same arguments as in the proof of Theorem 1, one finds that the least value of $|(X+c)Y|$ occurs when $u=v=1$. Hence the best possible constant for Theorem 2 cannot be less than
$$\frac{\frac{1}{2}(1+\theta)(1-\phi)}{\theta+\phi}=\frac{\frac{1}{2}(11\cdot 099\ldots)(0\cdot 382\ldots)}{10\cdot 717\ldots}=\frac{1}{5\cdot 05\ldots}.$$

ON THE CONVERGENCE OF EIGENFUNCTION EXPANSIONS

By E. C. TITCHMARSH (*Oxford*)

[Received 3 May 1951]

1. THE problem considered is that of the differential equation

$$\frac{d^2\phi}{dx^2}+\{\lambda-q(x)\}\phi = 0 \quad (0 \leqslant x < \infty) \tag{1.1}$$

with the boundary condition

$$\phi(0)\cos\alpha+\phi'(0)\sin\alpha = 0,$$

it being assumed that there are discrete eigenvalues $\lambda_0, \lambda_1,..., \lambda_n \to \infty$, with eigenfunctions $\psi_n(x)$. In a previous paper* I showed that the expansion of an arbitrary function $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{1.2}$$

where $$c_n = \int\limits_0^\infty f(t)\psi_n(t)\,dt,$$

is summable in the sense that

$$\lim_{\Lambda\to\infty} \sum_{\lambda_n<\Lambda} \left(1-\frac{\lambda_n}{\Lambda}\right)c_n \psi_n(x) = f(x),$$

provided that $f(x)$ is $L^2(0,\infty)$, and that, in the neighbourhood of the point x, a condition (e.g. continuity) similar to that required for the summability $(C,1)$ of an ordinary Fourier series, is satisfied.

In the present paper I consider the problem of the convergence in the ordinary sense of the expansion (1.2). In the theory of Fourier series we can pass from summability to convergence merely by imposing a more stringent condition on the function $f(t)$ in the neighbourhood of the point $t = x$. In the theory considered here this does not seem to be sufficient, and other assumptions have to be made, restricting the functions $q(x)$. The result is similar to that given in Ch. IX of my book *Eigenfunction Expansions*, but it is more general, since it is not restricted to the case of functions of bounded variation considered there.

THEOREM. *Let $q(x)$ be twice differentiable,*

$$q'(x) > 0, \qquad q''(x) \geqslant 0, \qquad q''(x) \leqslant \{q'(x)\}^\gamma$$

for sufficiently large values of x, where $1 < \gamma < \frac{4}{3}$. Let $f(t)$ be $L^2(0,\infty)$,

* *Quart. J. of Math.* (Oxford) (2), 2 (1951), 258–68.

Quart. J. Math. Oxford (2), 3 (1952), 139–44

and, in the neighbourhood of $t = x$, let $f(t)$ satisfy any condition which is sufficient for the convergence of an ordinary Fourier series. Then (1.2) *holds in the sense of ordinary convergence.*

The conditions imposed on $q(x)$ are satisfied, for example, if $q(x) = x^c$, where $c > 1$.

2. I recall the following results obtained in my book. Let p_n be defined by the equation $q(p_n) = \lambda_n$. Then under the above conditions

$$n \sim \frac{1}{\pi}\int\limits_0^{p_n} \{\lambda_n - q(x)\}^{\frac{1}{2}}\,dx,$$

so that in particular

$$n = O(p_n \lambda_n^{\frac{1}{2}}). \tag{2.1}$$

If $N(\lambda)+1$ denotes the number of eigenvalues $\lambda_0, \lambda_1,\ldots$ not exceeding λ, then (Theorem 7.6)

$$N(\lambda+\surd\lambda) = N(\lambda)\{1+O(\lambda^{-\frac{1}{2}})\}. \tag{2.2}$$

For any fixed x, by Lemma 9.8,

$$\psi_n(x) = O(p_n^{-\frac{1}{2}}). \tag{2.3}$$

Now let $f(t)$ be any function of $L^2(0,\infty)$. Let ϵ be a given positive number. Then by well-known constructions in the theory of the Lebesgue integral, we can express $f(t)$ in the form

$$f(t) = g(t)+h(t),$$

where $g(t)$ and $q(t)g(t)-g''(t)$ are $L^2(0,\infty)$,

$$g(0)\cos\alpha+g'(0)\sin\alpha = 0,$$

$g(x) = 0$, and $g(t) = 0$ for t sufficiently large, while

$$\int\limits_0^\infty \{h(t)\}^2\,dt < \epsilon^2.$$

Let
$$a_n = \int\limits_0^\infty \psi_n(x)g(x)\,dx, \qquad b_n = \int\limits_0^\infty \psi_n(x)h(x)\,dx,$$

so that $a_n+b_n = c_n$. By Theorem 2.7 (i) of my book,

$$A_N = \sum_{n=0}^{N} a_n\psi_n(x) \to g(x) = 0.$$

It is therefore sufficient to prove that, if

$$B_N = \sum_{n=0}^{N} b_n\psi_n(x),$$

then

$$|B_N - f(x)| < A\epsilon$$

for N large enough; for then

$$\left|f(x)-\sum_{n=0}^{N} c_n\psi_n(x)\right| = |f(x)-A_N-B_N|$$
$$\leqslant |f(x)-B_N|+|A_N| < A\epsilon$$

for N large enough.

3. Let $\theta(x,\lambda)$, $\phi(x,\lambda)$, and $\psi(x,\lambda)$ have the same meanings as in the previous paper, and let

$$\Phi(x,\lambda) = \Phi(x,\lambda,h) = \psi(x,\lambda)\int_0^x \phi(\xi,\lambda)h(\xi)\,d\xi+\phi(x,\lambda)\int_x^\infty \psi(\xi,\lambda)h(\xi)\,d\xi.$$

Let
$$J = \frac{1}{2\pi i}\int \Phi(x,\lambda)\,d\lambda,$$

where the contour in the λ-plane is symmetrical about the real axis, and its upper half corresponds to the quarter-square $(K,\ K+iK,\ iK)$ in the plane of $\kappa = \sqrt{\lambda}$, where $K = \sqrt{\lambda_n}+\frac{1}{8}$. It is assumed that no pole of $\Phi(x,\lambda)$ lies actually on the contour: if it did, a slightly different contour could obviously be chosen. We have then

$$J = \sum_{\lambda_m<(\sqrt{\lambda_n}+\frac{1}{8})^2} b_m\psi_m(x).$$

Also, as $n\to\infty$,
$$J = \sum_{m\leqslant n} b_m\psi_m(x)+o(1); \tag{3.1}$$

for the difference between these two sums is

$$\left|\sum_{\lambda_n<\lambda_m<(\sqrt{\lambda_n}+\frac{1}{8})^2} b_m\psi_m(x)\right| \leqslant \left\{\sum_{\lambda_n<\lambda_m<(\sqrt{\lambda_n}+\frac{1}{8})^2} b_m^2 \sum_{\lambda_n<\lambda_m<(\sqrt{\lambda_n}+\frac{1}{8})^2} \psi_m^2(x)\right\}^{\frac{1}{2}}.$$

Since
$$(\sqrt{\lambda_n}+\tfrac{1}{8})^2 = \lambda_n+\tfrac{1}{4}\sqrt{\lambda_n}+\tfrac{1}{64} < \lambda_n+\sqrt{\lambda_n}$$

if λ_n is large enough, the last sum contains not more than

$$N(\lambda_n+\sqrt{\lambda_n})-N(\lambda_n) = O\{\lambda_n^{-\frac{1}{2}}N(\lambda_n)\} = O(n\lambda_n^{-\frac{1}{2}}) = O(p_n)$$

terms, each of which is $O(p_n^{-1})$. This sum is therefore bounded. Also $\sum b_m^2$ is convergent by the Bessel inequality, so that the first sum on the right is $o(1)$. This proves (3.1).

Let $\kappa = \sqrt{\lambda} = \sigma+i\tau$. Then the analysis of the previous paper, with $h(\xi)$ instead of $f(\xi)$, shows that for $\sigma>0$ and $-1\leqslant\tau\leqslant 1$,

$$\Phi(x,\kappa^2) = O\left(\frac{\epsilon}{\sigma|\tau|}\right).$$

Let
$$\Psi(x,\lambda) = \sum_{\lambda_n<\lambda_m<\lambda_n+\sqrt{\lambda_n}} \frac{b_m\psi_m(x)}{\lambda-\lambda_m}.$$

Then, if $\lambda = \mu+i\nu$

$$|\Psi(x,\lambda)| \leqslant \frac{1}{|\nu|} \sum_{\lambda_n<\lambda_m<\lambda_n+\surd\lambda_n} |b_m \psi_m(x)| < \frac{A\epsilon}{|\nu|}$$

by an argument similar to the above, since

$$\sum_{m=0}^{\infty} b_m^2 \leqslant \int_0^\infty \{h(x)\}^2\,dx < \epsilon^2.$$

Hence, for $\sigma > 0$, $$\Psi(x,\kappa^2) = O\left(\frac{\epsilon}{\sigma|\tau|}\right).$$

Let $$\Omega(x,\kappa^2) = \Phi(x,\kappa^2) - \Psi(x,\kappa^2).$$

Then, for $\sigma > 0$ and $-1 \leqslant \tau \leqslant 1$,

$$\Omega(x,\kappa^2) = O\left(\frac{\epsilon}{\sigma|\tau|}\right).$$

Also $\Omega(x,\lambda)$ is regular for real values of λ such that $\lambda_n < \lambda < \lambda_n+\surd\lambda_n$, and so $\Omega(x,\kappa^2)$ is regular for real values of κ such that

$$\surd\lambda_n < \kappa < \surd(\lambda_n+\surd\lambda_n).$$

Since $$\surd(\lambda_n+\surd\lambda_n) > \surd\lambda_n+\tfrac{1}{4}$$

if λ_n is large enough, it follows that $\Omega(x,\kappa^2)$ is regular for real values of κ such that

$$\surd\lambda_n < \kappa < \surd\lambda_n+\tfrac{1}{4},$$

and so throughout the strip

$$\surd\lambda_n < \sigma < \surd\lambda_n+\tfrac{1}{4}.$$

It then follows from Lemma 2.11 of my book that

$$\Omega(x,\kappa^2) = O(\epsilon/\sigma)$$

for $\sigma = \surd\lambda_n+\frac{1}{8} = K$ and $-1 \leqslant \tau \leqslant 1$. Hence

$$\int_{K-i}^{K+i} \Omega(x,\kappa^2)\kappa\,d\kappa = \int_{-1}^{1} O\left(\frac{\epsilon}{K}\right)K\,d\tau = O(\epsilon).$$

Also

$$2\int_{K-i}^{K+i} \Psi(x,\kappa^2)\kappa\,d\kappa = \int_{(K-i)^2}^{(K+i)^2} \sum_{\lambda_n<\lambda_m<\lambda_n+\surd\lambda_n} \frac{b_m\psi_m(x)}{\lambda-\lambda_m}\,d\lambda$$

$$= \sum_{\lambda_n<\lambda_m<\lambda_n+\surd\lambda_n} b_m\psi_m(x) \int_{(K-i)^2}^{(K+i)^2} \frac{d\lambda}{\lambda-\lambda_m}$$

$$= O\left\{\sum_{\lambda_n<\lambda_m<\lambda_n+\surd\lambda_n} |b_m\psi_m(x)|\right\} = O(\epsilon)$$

as before. Altogether

$$\int_{K-i}^{K+i} \Phi(x,\kappa^2)\kappa\,d\kappa = O(\epsilon).$$

4. On the remainder of the contour we write

$$\begin{aligned}\Phi(x,\lambda) = {} & \psi(x,\lambda)\int_0^{x-\delta} \phi(\xi,\lambda)h(\xi)\,d\xi + \\ & +\psi(x,\lambda)\int_{x-\delta}^{x} \phi(\xi,\lambda)h(\xi)\,d\xi + \\ & +\phi(x,\lambda)\int_{x}^{x+\delta} \psi(\xi,\lambda)h(\xi)\,d\xi + \\ & +\phi(x,\lambda)\int_{x+\delta}^{\infty} \psi(\xi,\lambda)h(\xi)\,d\xi \\ = {} & \Phi_1+\Phi_2+\Phi_3+\Phi_4,\end{aligned}$$

where $f(\xi)$, and so also $h(\xi)$ satisfies conditions for the convergence of a Fourier series in $x-\delta < \xi < x+\delta$. Let

$$\frac{1}{\pi i}\left(\int_{-iK}^{K-iK} + \int_{K-iK}^{K-i} + \int_{K+i}^{K+iK} + \int_{K+iK}^{iK}\right)\Phi(x,\kappa^2)\kappa\,d\kappa = J_1+J_2+J_3+J_4$$

corresponding to $\Phi_1,\ldots, \Phi_4$.

As in the previous paper, for $|\tau| > 1$,

$$\Phi_1 = O(\epsilon\sigma^{-1}e^{-\delta\tau}),$$

and, for $0 \leqslant \sigma \leqslant 1$ and τ large enough,

$$\Phi_1 = O(\epsilon e^{-\delta\tau}).$$

Hence, δ being fixed,

$$J_1 = O\left(\int_1^K \epsilon\frac{e^{-\delta\tau}}{K}K\,d\tau\right)+O\left(\int_0^K \epsilon e^{-\delta K}\,d\sigma\right) = O(\epsilon).$$

A similar argument applies to J_4.

For Φ_3, we have the same approximate formulae as in the previous paper, except that now a factor ϵ appears in those terms whose upper

bounds involve $\int \{h(\xi)\}^2\,d\xi$. It follows that the upper half of the contour contributes to J_3

$$\frac{1}{2\pi i}\int\limits_{K+i}^{iK}\frac{e^{-i\kappa x}}{2i\kappa}\left\{\int\limits_{x}^{x+\delta}e^{i\kappa\xi}h(\xi)\,d\xi\right\}2\kappa\,d\kappa+$$

$$+O\left\{\int\limits_{1}^{K}\left(\frac{1}{K^2}\int\limits_{x}^{x+\delta}|h(\xi)|\,d\xi+\frac{\epsilon e^{-2x\tau}}{K}+\frac{\epsilon e^{-(2b-2x-\delta)\tau}}{K}\right)K\,d\tau\right\}+O\left(\int\limits_{0}^{e^{-K}}K\,d\sigma\right)+$$

$$+O\left\{\int\limits_{e^{-K}}^{K}\left(\frac{1}{K^2}\int\limits_{x}^{x+\delta}|h(\xi)\,d\xi|+\frac{\epsilon e^{-2xK}}{K}+\frac{\epsilon e^{-(2b-2x-\delta)K}}{\sigma}\right)K\,d\sigma\right\}.$$

The O-terms give

$$O\left(\int\limits_{x}^{x+\delta}|h(\xi)|\,d\xi\right)+O(\epsilon)+o(1) = O(\epsilon)+o(1).$$

In the main term we can replace the lower limit of integration by K with error

$$\frac{1}{2\pi i}\int\limits_{K}^{K+i}\frac{e^{-i\kappa x}}{2i\kappa}\left\{\int\limits_{x}^{x+\delta}e^{i\kappa\xi}h(\xi)\,d\xi\right\}2\kappa\,d\kappa = \int\limits_{0}^{1}O(\epsilon)\,d\tau = O(\epsilon).$$

Also

$$\frac{1}{2\pi i}\int\limits_{K}^{iK}\frac{e^{-i\kappa x}}{2i\kappa}\left\{\int\limits_{x}^{x+\delta}e^{i\kappa\xi}h(\xi)\,d\xi\right\}2\kappa\,d\kappa = -\frac{1}{2\pi}\int\limits_{x}^{x+\delta}h(\xi)\,d\xi\int\limits_{K}^{iK}e^{i\kappa(\xi-x)}\,d\kappa$$

$$= -\frac{1}{2\pi}\int\limits_{x}^{x+\delta}h(\xi)\frac{e^{-K(\xi-x)}-e^{iK(\xi-x)}}{i(\xi-x)}\,d\xi.$$

Since the lower half of the contour contributes the conjugate of this the two together give

$$\frac{1}{\pi}\int\limits_{x}^{x+\delta}h(\xi)\frac{\sin\{K(\xi-x)\}}{\xi-x}\,d\xi.$$

A similar integral with limits $(x-\delta, x)$ arises from J_2, so that altogether we obtain

$$\frac{1}{\pi}\int\limits_{0}^{\delta}\{h(x+t)+h(x-t)\}\frac{\sin Kt}{t}\,dt+O(\epsilon)+o(1).$$

This integral is the well-known expression which occurs in the theory of Fourier series, and the result therefore follows.

A NOTE ON A THEOREM OF PÓLYA'S

By R. WILSON (*Swansea*)

[Received 30 May 1951]

1. Introduction

In a noteworthy paper† Pólya has obtained the result below.

THEOREM 1. *Let $f(z)$ be an analytic function and let*

$$\lim_{n\to\infty} |b_n|^{1/n} = 0. \tag{1}$$

Then

$$f^*(z) = \sum_0^\infty (-1)^\nu \frac{b_\nu}{\nu!} f^{(\nu)}(z) \tag{2}$$

is a regular analytic function in the entire region of existence of $f(z)$ and every 'zugängliche' singularity of $f(z)$ is also singular for $f^*(z)$.

Pólya has defined the *zugängliche* (approachable) singularity as follows: let $f(z)$ have such a singularity at the point $z = \alpha$ and consider a sufficiently small circular region centred on α. Take any diameter dividing the circle into two open semicircular regions and let the diameter be divided into two radii without end-points. Then, if it is possible so to choose the diameter that $f(z)$ is regular in at least one open semicircle and along at least one of the two bounding radii, the point α is said to be a *zugängliche* singularity.

It will be noted that this definition is in no way related to any particular circle of convergence. There is, however, a certain convenience in classifying singularities in relation to the circle of convergence, as Pólya himself has done‡ except in the case of the *zugängliche* singularity. I therefore give an appropriate definition for a singular point on the circle of convergence which will include as special cases those defined by Pólya under the names *gut zugängliche*, *fastisoliert*, and *isoliert* (easily approachable, almost isolated, and isolated).‡

Suppose that $f(z)$ can be developed in a series of powers of z (or of z^{-1}) with a finite non-zero radius of convergence. A point $z = \alpha$ on the circle of convergence of $f(z)$ is said to be *isolated with regard to the circumference* if a positive number δ exists such that there is no other singularity of $f(z)$ on the arc $|\arg z - \arg\alpha| < \delta$ of the circle of convergence. For brevity I shall describe such a singularity as *unique*.

Neither of these definitions includes the other. For example, the

† (7).

‡ (9) § 69.

point of contact of a line of singularities with the circle of convergence is not a *zugängliche* singularity although it is unique. If we replace the line by a curve of singularities, then the point of contact with the circle of convergence is a unique singularity but is *zugängliche* only if the curve is convex to the origin either above or below the point of contact. On the other hand, the end-point of a singular arc may be a *zugängliche* singularity although it is clearly not unique. Nevertheless, each of these types of singularities contains as special cases those defined by Pólya and referred to previously.†

I prove the following theorem relating to a unique singularity on the circle of convergence.

THEOREM 2. *Let $f(z)$ have a finite non-zero radius of convergence. Then every singular point of $f(z)$ isolated with regard to the circumference is a unique singularity of $f^*(z)$, defined by* (1) *and* (2), *while $f^*(z)$ is regular at every regular point of $f(z)$ on its circle of convergence.*

Although, in general, we cannot know that $f(z)$ and $f^*(z)$ have the same circle of convergence, if $f(z)$ has even *one* singularity isolated with regard to the circumference, then this information follows.

In so far as the idea of the unique singularity is associated with a particular circle of convergence, it is not as general as that of the *zugängliche* singularity. In the same sense Theorem 2 is not so wide as Theorem 1. Nevertheless, the results of Theorem 2 arise quite naturally from the known properties of integral functions of order 1 and, as it is convenient to consider these as functions regular in an angle, we are able to extend the results to functions defined by irregular power series.

In § 2, I analyse the content and implications of Theorem 2 and deduce a correlative theorem for integral functions of order 1. This is proved for integral functions of finite order ρ in § 3 and generalizations to quasi-regular functions and quasi-integral functions are given in § 4.

2. Analysis of Theorems

Let
$$f(z) = \sum_0^\infty a_n z^{-n-1}, \qquad g(z) = \sum_0^\infty b_n z^{-n-1}, \tag{3}$$
where the sequence (b_n) satisfies (1), so that $g(z)$ is regular in the whole plane save at the origin and is zero at infinity.

It is not difficult to show that $f^*(z)$ defined by (2) is the Hurwitz composition function‡
$$\frac{1}{2\pi i}\int_C f(u)g(z-u)\,du \tag{4}$$

† (9) § 69.

‡ As in (7).

of $f(z)$ and $g(z)$. In accordance with the Hurwitz–Pincherle composition theory† the possible singularities of $f^*(z)$ have their affixes at the sums of the affixes of the singularities of $f(z)$ and $g(z)$. Hence the set of singularities of $f^*(z)$ must be a subset of the set of singularities of $f(z)$ and its closure unless it is the set itself. This only tells us where the set of singularities of $f^*(z)$ *may* lie, but Theorems 1 and 2 tell us where they *must* lie if the associated singularities of $f(z)$ are *zugängliche* or unique as the case may be.

Let $f(z)$ of (3) converge for $|z| > h$ and diverge for $|z| < h$. Then h is the type of its inverse transform

$$F(z) = \frac{1}{2\pi i}\int\limits_C f(\zeta)e^{z\zeta}\,d\zeta, \tag{5}$$

where C is a closed curve containing the circle $|z| = h$. Thus $F(z)$ is an integral function of order 1 and type h, while the inverse transform of $g(z)$

$$G(z) = \frac{1}{2\pi i}\int\limits_{C'} g(\zeta)e^{z\zeta}\,d\zeta, \tag{6}$$

where C' is a closed curve containing the origin, is an integral function of order 1 of minimum type, from (1).

Let $h(\phi)$ be the indicator‡ of $F(z)$. Then, defining a *direction of strongest growth* of $F(z)$ to be one along which $h(\phi)$ attains its maximum h, we have§

LEMMA 1. *If ϕ' is a direction of strongest growth of $F(z)$, then the radius* $\arg z = -\phi'$ *meets the circle of convergence of $f(z)$ in a singular point and conversely.*

A similar relation holds between $f^*(z)$ and its inverse transform

$$F^*(z) = \frac{1}{2\pi i}\int\limits_C f^*(\zeta)e^{z\zeta}\,d\zeta. \tag{7}$$

It follows that, to a unique singularity of $f^*(z)$ or $f(z)$, there corresponds a unique direction of strongest growth of $F^*(z)$ or $F(z)$ as the case may be. By a *unique* direction of strongest growth $\phi = \phi'$ we mean that, in a sufficiently small angle $|\arg z - \phi'| < \delta$ $(\delta > 0)$, there is no other direction of strongest growth.

Now relations (2) and (4) can be put in the form

$$f^*(z) = \sum_0^\infty \left(\sum_0^n \frac{a_\nu b_{n-\nu}}{\nu!(n-\nu)!}\right)\frac{n!}{z^{n+1}}, \tag{8}$$

† See (5) § 13 for further references.

‡ (1), (8) Kap. II, (10) §§ 5.7–5.81.

§ (8) 587–8.

and so it follows from (5), (6), and (7) that

$$F^*(z) = F(z)G(z). \tag{9}$$

Thus the proof of Theorem 2 depends on the following:

THEOREM 3. *The product of two integral functions of order* 1, *one of mean type and the other of minimum type, has the same unique directions of strongest growth as the former.*

3. Proof of Theorem 3

The proof of Theorem 3 depends on two lemmas. The first is due to Vl. Bernstein.† Let $F(z)$ be regular and of order ρ in an angle and let $h(\phi)$ be its indicator. Then we have

LEMMA 2. *For every ϕ inside the angle and any positive ϵ, the inequality*

$$\log|F(re^{i\phi})| > [h(\phi)-\epsilon]r^\rho$$

holds for a set of r of positive upper linear density at least.

The second is due to M. L. Cartwright.‡ Let $G(z)$ be regular and of order ρ of minimum type, in an angle greater than π/ρ. Then we have

LEMMA 3. *For every ϕ inside the angle and any positive ϵ', the inequality*

$$\log|G(re^{i\phi})| > -\epsilon' r^\rho$$

holds for a set of r of linear density one.

From (9) and Lemmas 1 and 2 it follows that

$$\log|F^*(re^{i\phi})| > [h(\phi)-\epsilon'']r^\rho \tag{10}$$

holds for a set of r of positive linear upper density at least, where ϵ'' (> 0) is arbitrary.

From (9) it follows that the type of $F^*(z)$ cannot exceed h. On the other hand, putting $\phi = \phi'$, a direction of strongest growth of $F(z)$, we have $h(\phi') \equiv h$ in (10), so that $\phi = \phi'$ is also a direction of strongest growth of $F^*(z)$. Hence we have

THEOREM 4. *The product of two integral functions of order ρ, one of mean type and the other of minimum type, has the same unique directions of strongest growth as the former.*

Theorem 3 is a special case of Theorem 4 with $\rho = 1$, and Theorem 2 follows at once.

4. Generalizations

Lemmas 2 and 3 hold for functions regular in an appropriate angle and will therefore apply to quasi-integral functions of exponential

† (**1**). ‡ (**2**) Th. 2. See also (**4**) 428 with $RS(R) = \epsilon R$.

type.† The Laplace transforms of such functions are called 'quasi-regular' functions and they may be represented by irregular power series of the form‡

$$f(z) = \sum_0^\infty a_n z^{-\lambda_n - 1}, \qquad g(z) = \sum_0^\infty b_n z^{-\lambda'_n - 1}, \tag{11}$$

where (λ_n), (λ'_n) are monotone sequences of positive numbers, each tending to infinity.

Corresponding to (8), the generalized Hurwitz composition function may be written§

$$f^*(z) = \sum_{m,n=0}^{\infty} \frac{\Gamma(\lambda_m+\lambda'_n+1)a_m b_n}{\Gamma(\lambda_m+1)\Gamma(\lambda'_n+1)z^{\lambda_m+\lambda'_n+1}}. \tag{12}$$

These three functions may be represented on a logarithmic surface about the origin, for they are made uniform by the transformation $z = e^w$.

A unique singularity for such functions is then defined as in § 1 as a singularity isolated with regard to the periphery of holomorphy of the function concerned. Directions of strongest growth for the inverse transforms of (11) and (12), $F(z)$, $G(z)$, and $F^*(z)$ are defined as in § 2, and the relations between the directions of strongest growth of these quasi-integral functions and the singular directions of their transforms in (11) and (12) are as stated in Lemma 1.||

Further, as in (9), $\qquad F^*(z) = F(z)G(z).$

If $G(z)$ is of minimum type, so that condition (1) holds, then, since each of these functions is regular in any angle, however large, we can apply Lemmas 2 and 3 to obtain the following:

THEOREM 5. *The product of two quasi-integral functions of order* 1, *one of mean type and the other of minimum type, has the same unique directions of strongest growth as the former.*

Finally, by processes similar to those of §§ 2 and 3 we are led to

THEOREM 6. *Every singular point isolated with regard to the periphery of holomorphy of* $f(z)$ *of* (11) *is a unique singularity of* $f^*(z)$ *of* (12) *and* $f^*(z)$ *is regular at every regular point of* $f(z)$ *on its periphery of holomorphy.*

The last parts of Theorem 2 and Theorem 6 follow because the indicators of $f(z)$ and $f^*(z)$ being the same, such indicators attain their maxima for the same directions.

† (6) §§ 6–11. ‡ (6) is largely concerned with this problem.

§ (3). The corresponding result on the composition of singularities is here given.

|| (6) Satz X.

By use of a theorem due to Pfluger† the results of Theorem 5 can easily be extended to quasi-integral functions of order ρ.

† (6) Satz XIII.

REFERENCES

1. Vl. Bernstein, *Ann. R. Sc. Normale Sup. di Pisa* (Sc. Fis. e Mat.), 2 (1933), 381–400.
2. M. L. Cartwright, *Proc. London Math. Soc.* (2) 38 (1935), 158–79.
3. H. G. Eggleston, *Proc. Cambridge Phil. Soc.* 47 (1951), 477–82.
4. A. J. Macintyre and R. Wilson, *Proc. London Math. Soc.* (2) 47 (1942), 404–35.
5. S. Mandelbrojt, *Les Singularités des Fonctions Analytiques Représentées par une Série de Taylor* (Paris, 1932).
6. A. Pfluger, *Comm. Math. Helvetici* 8 (1935), Part II, 3–43.
7. G. Pólya, *Göttinger Nachrichten* (1927) 187–95.
8. —— *Math. Zeits.* 29 (1929), 549–640.
9. —— *Annals of Math.* 34 (1933), 731–77.
10. E. C. Titchmarsh, *The Theory of Functions* (Oxford, 1932).

SOME DETERMINANTS WITH HYPERGEOMETRIC ELEMENTS

By J. L. BURCHNALL (*Durham*)

[Received 1 July 1951]

1. In a recent paper (**1**) I have shown that, if in any homogeneous isobaric invariant $I(a_0, a_1,..., a_n) \equiv I(a_r)$ of the binary n-ic $\sum \binom{n}{r} a_r X^{n-r} Y^r$ the coefficient a_r is replaced by

$$U_r^\lambda(x) = \binom{r+2\lambda-1}{r}^{-1} P_r^\lambda(x), \tag{1}$$

where $P_r^\lambda(x)$ is the ultraspherical polynomial of degree r and parameter λ, then

$$I\{U_r^\lambda(x)\} = C(\lambda)(x^2-1)^{\frac{1}{2}w}, \tag{2}$$

where $C(\lambda)$ is a constant and w is the weight of the invariant. If we apply this principle to the determinant of order $n+1$ which is known as the 'catalecticant' of the $2n$-ic, we obtain:

If $b_{rs} = U_{r+s}^\lambda(x)$ $(r, s = 0, 1,..., n)$, *then*

$$D_n^\lambda(x) \equiv |b_{rs}| = C(\lambda)(x^2-1)^{\frac{1}{2}n(n+1)}. \tag{3}$$

The constant $C(\lambda)$ has recently been expressed by Beckenbach and others [(**2**) 1] in the form of a multiple integral. I prove here the explicit formula

$$C(\lambda) = 2^{n(n+1)} \prod_{r=1}^{n} \frac{r!\,(\lambda)_r(\lambda)_r}{(2\lambda)_{2r}(2\lambda+r-1)_r} = 2^{-n(n+1)} \prod_{r=1}^{n} \frac{r!\,(2\lambda-1)_r}{(\lambda+\frac{1}{2})_r(\lambda-\frac{1}{2})_r} \tag{4}$$

where, as always, $(\kappa)_r = \kappa(\kappa+1)...(\kappa+r-1)$ and it is assumed that λ does not take values leading to zero factors in the denominators. Thus, for instance, if $\lambda = \frac{1}{2}$, we may employ the first form in (4) but not the second. By doing so we obtain $C(\frac{1}{2}) = 2^{-n^2}$ in agreement with (**1**) 12 (i).

Two lines of approach are possible: the first, which is somewhat indirect, introduces other determinants with hypergeometric elements which may be of interest: the second, which is more immediate, depends upon the identity of two bilinear forms and is given in § 6.

2. LEMMA

If $\{\phi_r(t)\}, \{\psi_r(t)\}$ $(r = 0, 1, 2,...)$ *are two sequences of polynomials satisfying the conditions*

$$\phi_0(t) = \text{constant}, \qquad \psi_0(t) = \text{constant},$$

$$\frac{d\phi_r}{dt} = r\phi_{r-1}, \qquad \frac{d\psi_r}{dt} = r\psi_{r-1}, \quad \textit{for all } r \geqslant 1,$$

Quart. J. Math. Oxford (2), 3 (1952), 151–7.

and $$\theta_s(t) = \sum_{r=0}^{n} \frac{(-n)_r}{r!}\phi_{n+s-r}(t)\psi_r(t) \quad (s = 0, 1, 2,...),$$

then $$\theta_0(t) = \text{constant}\dagger = c_0, \qquad \frac{d\theta_s}{dt} = s\theta_{s-1} \quad (s = 1, 2,...). \tag{5}$$

For

$$\frac{d\theta_s}{dt} = \sum_{r=0}^{n} \frac{(n+s-r)(-n)_r}{r!}\phi_{n+s-r-1}\psi_r + \sum_{r=1}^{n} \frac{(-n)_r}{(r-1)!}\phi_{n+s-r}\psi_{r-1}$$

$$= \sum_{r=0}^{n} \frac{(n+s-r)(-n)_r}{r!}\phi_{n+s-r-1}\psi_r + \sum_{r=0}^{n-1} \frac{(r-n)(-n)_r}{r!}\phi_{n+s-r-1}\psi_r$$

$$= s\sum_{r=0}^{n} \frac{(-n)_r}{r!}\phi_{n+s-1-r}\psi_r = s\theta_{s-1},$$

and the lemma is proved.

It follows from (5) by repeated integration that

$$\theta_s(t) = \sum_{r=0}^{s} \binom{s}{r} c_r t^{s-r}, \tag{6}$$

where c_r is the constant term in $\theta_r(t)$.

Take now, as functions evidently satisfying the conditions of the lemma,

$$\left.\begin{aligned} \phi_r(t) &= t^r {}_2F_1(a, -r; b; t^{-1}) \\ \psi_r(t) &= t^r {}_2F_1(1-a-n, -r; 2-b-2n; t^{-1}) \end{aligned}\right\} \tag{7}$$

where ${}_2F_1$ is the hypergeometric function. On picking out the constant term in $\theta_s(t)$ we find

$$(-1)^{n+s}c_s = \sum_{r=0}^{n} \frac{(-n)_r(a)_{n+s-r}(1-a-n)_r}{r!(b)_{n+s-r}(2-b-2n)_r}$$

$$= \frac{(a)_{n+s}}{(b)_{n+s}}\,{}_3F_2\left[\begin{matrix}1-a-n,\ 1-b-s-n,\ -n; \\ 1-a-s-n,\ 2-b-2n\end{matrix}\ 1\right]$$

$$= \frac{(a)_{n+s}(b-a)_n(-s)_n}{(b)_{n+s}(1-a-s-n)_n(b+n-1)_n}$$

$$= \begin{cases} 0, & \text{if} \quad s < n \\ \dfrac{n!(a)_n(b-a)_n}{(b)_{2n}(b+n-1)_n}, & \text{if} \quad s = n; \end{cases}$$

where in the reduction we have used Saalschütz's theorem.

† As I have already shown in (**1**).

Hence, with ϕ_r, ψ_r defined by (7), we have

$$\left.\begin{aligned} \theta_s(t) &\equiv 0, \quad \text{if} \quad 0 \leqslant s < n \\ \theta_n(t) &\equiv \frac{n!(a)_n(b-a)_n}{(b)_{2n}(b+n-1)_n} \end{aligned}\right\}. \tag{8}$$

3. Consider now the determinant

$$\Delta_n(x,a,b) = |{}_2F_1(a,-r-s;b;x)| \quad (r,s = 0,1,\dots,n)$$

where b may not take integral values. If on this determinant we perform the transformation indicated by

$$(\operatorname{col} n)' = \sum_{r=0}^{n} \frac{(-n)_r}{r!}\,{}_2F_1(1-a-n,-r;2-b-2n;x)\operatorname{col}(n-r),$$

we shall, by (8), have zeros in the last column in all places except the lowest, and in this we obtain

$$\frac{n!(a)_n(b-a)_n}{(b)_{2n}(b+n-1)_n}x^{2n}.$$

Hence $$\Delta_n(x,a,b) = x^{2n}\frac{n!(a)_n(b-a)_n}{(b)_{2n}(b+n-1)_n}\Delta_{n-1}(x,a,b)$$

and, since $\Delta_0(x,a,b) = 1$,

$$\Delta_n(x,a,b) = x^{n(n+1)}\prod_{r=1}^{n}\frac{r!(a)_r(b-a)_r}{(b)_{2r}(b+r-1)_r}. \tag{9}$$

We observe that $$\Delta_n(x,b-a,b) = \Delta_n(x,a,b). \tag{10}$$

Take now $x = 1$ and recall that

$${}_2F_1(b-a,-r-s;b;1) = \frac{(a)_{r+s}}{(b)_{r+s}}.$$

Then, from (9) and (10),

$$\left|\frac{(a)_{r+s}}{(b)_{r+s}}\right| = \prod_{r=1}^{n}\frac{r!(a)_r(b-a)_r}{(b)_{2r}(b+r-1)_r}. \tag{11}$$

4. Let Δ_n^r be the n-rowed minor of $\Delta_n(x,a,b)$ obtained by deleting the last row and the $(n-r)$th column, so that

$$\Delta_n^0 = \Delta_{n-1}(x,a,b). \tag{12}$$

The equations

$$\sum_{r=0}^{n}(-1)^r\,{}_2F_1(a,-n-s+r;b;x)X_r = 0 \quad (s = 0,1,\dots,n-1)$$

have the solutions

$$X_r = \Delta_n^r, \qquad X_r = \binom{n}{r}{}_2F_1(1-a-n,-r;2-b-2n;x)$$

which are not in general independent. Hence, using (12),

$$\Delta_n^r = \binom{n}{r} x^{n(n-1)}{}_2F_1(1-a-n, -r; 2-b-2n; x) \times \prod_{r=1}^{n-1} \frac{r!(a)_r(b-a)_r}{(b)_{2r}(b+r-1)_r}. \tag{13}$$

5. We are now in a position to establish (4). If

$$2u = x+1, \qquad 2v = x-1,$$

then†

$$U_n^\lambda(x) = u^n {}_2F_1\left[\begin{matrix}-n, & -n-\lambda+\frac{1}{2}; & \frac{v}{u} \\ & \lambda+\frac{1}{2} & \end{matrix}\right] = \theta_n(u, v) \tag{14}$$

and, from (3),

$$|\theta_{r+s}(u, v)| = 2^{n(n+1)} C(\lambda)(uv)^{\frac{1}{2}n(n+1)}. \tag{15}$$

Both sides of (15) are homogeneous of degree $n(n+1)$ in u, v, and (15) is an identity in these variables independent of the particular values of u, v in (14). We may determine $C(\lambda)$ by letting $u = v = 1$. But

$$\theta_{r+s}(1, 1) = \frac{(2\lambda+r+s)_{r+s}}{(\lambda+\frac{1}{2})_{r+s}} = 2^{2r+2s} \frac{(\lambda)_{r+s}}{(2\lambda)_{r+s}}$$

and

$$|\theta_{r+s}(1, 1)| = 2^{2n(n+1)} \left| \frac{(\lambda)_{r+s}}{(2\lambda)_{r+s}} \right| = 2^{2n(n+1)} \prod_{r=1}^{n} \frac{r!(\lambda)_r(\lambda)_r}{(2\lambda)_{2r}(2\lambda+r-1)_r}$$

by (11).

Comparison with (15) gives

$$\begin{aligned} C(\lambda) &= 2^{n(n+1)} \prod_{r=1}^{n} \frac{r!(\lambda)_r(\lambda)_r}{(2\lambda)_{2r}(2\lambda+r-1)_r} \\ &= \prod_{r=1}^{n} \frac{r!(\lambda)_r}{(\lambda+\frac{1}{2})_r(2\lambda+r-1)_r} \\ &= 2^{-n(n+1)} \prod_{r=1}^{n} \frac{r!(2\lambda-1)_r}{(\lambda+\frac{1}{2})_r(\lambda-\frac{1}{2})_r}. \end{aligned}$$

6. An alternative method of establishing (4) and (9) is to reduce to canonical form the corresponding quadratic or bilinear forms. The identities mentioned are in fact immediate consequences of the following, which have an interest of their own:

$$\sum_{s=0}^{n} \sum_{t=0}^{n} U_{s+t}^\lambda(x) X_s Y_t$$

$$= \sum_{r=0}^{n} \left(\frac{x^2-1}{4}\right)^r \frac{r!(2\lambda-1)_r}{(\lambda+\frac{1}{2})_r(\lambda-\frac{1}{2})_r} \left\{\sum_{s=r}^{n} \binom{s}{r} U_{s-r}^{\lambda+r}(x) X_s\right\} \left\{\sum_{t=r}^{n} \binom{t}{r} U_{t-r}^{\lambda+r}(x) Y_t\right\} \tag{16}$$

† (1), § 1 (4).

and

$$\sum_{s=0}^{n}\sum_{t=0}^{n} {}_2F_1\begin{bmatrix} a, & -s-t; & x \\ & b & \end{bmatrix} X_s Y_t$$

$$= \sum_{r=0}^{n} x^{2r} \frac{r!(a)_r(b-a)_r}{(b)_{2r}(b+r-1)_r} \left\{\sum_{s=r}^{n} \binom{s}{r} {}_2F_1\begin{bmatrix} a+r, -s+r; & x \\ b+2r & \end{bmatrix} X_s\right\} \times$$

$$\times \left\{\sum_{t=r}^{n} \binom{t}{r} {}_2F_1\begin{bmatrix} a+r, & -t+r; & x \\ b+2r & \end{bmatrix} Y_t\right\}. \quad (17)$$

The identity underlying (17) is

$${}_2F_1\begin{bmatrix} a, & -s-t; & x \\ & b & \end{bmatrix} = \sum_{r=0}^{\min(s,t)} \frac{(a)_r(b-a)_r(-s)_r(-t)_r}{r!(b)_{2r}(b+r-1)_r} x^{2r} \times$$

$$\times {}_2F_1\begin{bmatrix} a+r, & -s+r; & x \\ b+2r & \end{bmatrix} {}_2F_1\begin{bmatrix} a+r, & -t+r; & x \\ b+2r & \end{bmatrix},$$

which is a special case of a formula

$${}_2F_1\begin{bmatrix} a, & b+b'; & x \\ & c & \end{bmatrix} = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(b')_r(c-a)_r}{r!(c)_{2r}(c+r-1)_r} x^{2r} \times$$

$$\times {}_2F_1\begin{bmatrix} a+r, & b+r; & x \\ c+2r & \end{bmatrix} {}_2F_1\begin{bmatrix} a+r, & b'+r; & x \\ c+2r & \end{bmatrix}$$

given† some ten years ago by T. W. Chaundy and the author. It remains to establish (16) or the equivalent identity

$$U_{s+t}^{\lambda}(x) = \sum_{r=0}^{\min(s,t)} \left(\frac{x^2-1}{4}\right)^r \frac{(2\lambda-1)_r(-s)_r(-t)_r}{r!(\lambda+\frac{1}{2})_r(\lambda-\frac{1}{2})_r} U_{s-r}^{\lambda+r}(x) U_{t-r}^{\lambda+r}(x). \quad (18)$$

On setting‡ $$U_n^{\lambda}(x) = \sum_{r=0}^{[\frac{1}{2}n]} \frac{(-n)_{2r}}{r!(\lambda+\frac{1}{2})_r} x^{n-2r} \left(\frac{x^2-1}{4}\right)^r$$

everywhere in (18) we find that the result to be established is

$$\sum_{p,q,r}^{p+q+r=m} \frac{(2\lambda-1)_r(-s)_r(-t)_r(-s+r)_{2p}(-t+r)_{2q}}{r!(\lambda+\frac{1}{2})_r(\lambda-\frac{1}{2})_r p!\, q!(\lambda+\frac{1}{2}+r)_p(\lambda+\frac{1}{2}+r)_q}$$

$$= \frac{(-s-t)_{2m}}{m!(\lambda+\frac{1}{2})_m} \quad (2m < s+t). \quad (19)$$

Now the expression on the left is

$$\sum_{p,q,r} \frac{(2\lambda-1)_r(\lambda+\frac{1}{2})_r(-s)_{2p+r}(-t)_{2q+r}}{p!\, q!\, r!(\lambda-\frac{1}{2})_r(\lambda+\frac{1}{2})_{p+r}(\lambda+\frac{1}{2})_{q+r}}.$$

† (3), 114, 2 (16). ‡ (1), § 3.

In this set $2q+r = k$, $2p+r = 2m-k$. Collecting together the terms with the same k we find that, if $k \leqslant m$, they are

$$\sum_{q=0}^{2q\leqslant k} \frac{(-s)_{2m-k}(-t)_k(2\lambda-1)_k}{k!(m-k)!(\lambda-\frac{1}{2})_k(\lambda+\frac{1}{2})_m} \times$$

$$\times \frac{(\frac{3}{2}-\lambda-k)_{2q}(\frac{1}{2}-\lambda-m)_q(-k)_{2q}(\frac{1}{2}-\lambda-k)_q}{q!(2-2\lambda-k)_{2q}(\frac{1}{2}-\lambda-k)_{2q}(m+1-k)_q}$$

$$= \frac{(-s)_{2m-k}(-t)_k(2\lambda-1)_k}{k!(m-k)!(\lambda-\frac{1}{2})_k(\lambda+\frac{1}{2})_m} \times$$

$$\times {}_5F_4\left[\begin{matrix}\frac{1}{2}-\lambda-k,\ \frac{5}{4}-\frac{1}{2}\lambda-\frac{1}{2}k,\ \frac{1}{2}-\lambda-m,\ \frac{1}{2}-\frac{1}{2}k,\ -\frac{1}{2}k;\\ \frac{1}{4}-\frac{1}{2}\lambda-\frac{1}{2}k,\ m+1-k,\ 1-\lambda-\frac{1}{2}k,\ \frac{3}{2}-\lambda-\frac{1}{2}k\end{matrix}\ 1\right]$$

$$= \frac{(-s)_{2m-k}(-t)_k(2\lambda-1)_k}{k!(m-k)!(\lambda-\frac{1}{2})_k(\lambda+\frac{1}{2})_m} \times$$

$$\times \frac{\Gamma(m+1-k)\Gamma(1-\lambda-\frac{1}{2}k)\Gamma(\frac{1}{2}-\lambda-\frac{1}{2}k)\Gamma(m+\frac{1}{2})}{\Gamma(\frac{3}{2}-\lambda-k)\Gamma(1-\lambda)\Gamma(m+1-\frac{1}{2}k)\Gamma(m+\frac{1}{2}-\frac{1}{2}k)}$$

by a known formula,† the ${}_5F_4$ being well-poised, and on reduction the above expression is equal to

$$\frac{(2m)!(-s)_{2m-k}(-t)_k}{m!\,k!(2m-k)!(\lambda+\frac{1}{2})_m}.$$

If $k > m$, then we sum with respect to p instead of q, obtaining by a similar analysis the same final result. Hence the expression on the left of (19) is

$$\frac{1}{m!(\lambda+\frac{1}{2})_m}\sum_{k=0}^{2m}\frac{(2m)!(-s)_{2m-k}(-t)_k}{k!(2m-k)!} = \frac{(-s-t)_{2m}}{m!(\lambda+\frac{1}{2})_m},$$

by Vandermonde's theorem; (19) is therefore established and so, in consequence, are (18) and (16).

7. On setting $Y_t = X_t$ in (16) and (17) we see without difficulty that

(*a*) *If x is real, $|x| > 1$, and $\lambda > 0$, then*

$$\sum_{s=0}^{n}\sum_{t=0}^{n} U_{s+t}^{\lambda}(x)X_s X_t$$

is a positive definite quadratic form.‡

(*b*) *If x is real and not zero and $b > a > 0$, the quadratic form*

$$\sum_{s=0}^{n}\sum_{t=0}^{n} {}_2F_1\left[\begin{matrix}a,\ -s-t;\\ b\end{matrix}\ x\right]X_s X_t$$

is positive definite.

† (**4**), 27, 4.4 (1). ‡ Proved in (**2**) for $x > 1$, $\lambda > 0$.

By considering the special value $x = 1$ and writing $b-a$ for a we have

(c) *If $b > a > 0$ the quadratic form*

$$\sum_{s=0}^{n}\sum_{t=0}^{n}\frac{(a)_{s+t}}{(b)_{s+t}}X_s X_t$$

is positive definite.

In particular, setting $b = a+1$, we have

(d) *If $a > 0$ the quadratic form*

$$\sum_{s=0}^{n}\sum_{t=0}^{n}\frac{X_s X_t}{a+s+t}$$

is positive definite.

A simple alternative proof of (b) may be obtained by employing the Eulerian integral for the hypergeometric function. For

$$\sum_{s=0}^{n}\sum_{t=0}^{n} {}_2F_1\left[\begin{matrix}a, & -s-t; \\ & b\end{matrix}\, x\right]X_s X_t$$

$$= \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int\limits_0^1 u^{a-1}(1-u)^{b-a-1}\left\{\sum_{r=0}^{n}(1-xu)^r X_r\right\}^2 du,$$

which is evidently positive unless the final bracket vanishes for all u in $(0, 1)$. But, if $x \neq 0$, this implies

$$\sum_{r=k}^{n}\binom{r}{k}X_r = 0 \quad (k = 0, 1,..., n),$$

and so $X_r = 0$ for all r.

REFERENCES

1. J. L. Burchnall, 'An algebraic property of the classical polynomials', *Proc. London Math. Soc.* (3) 1 (1951).
2. E. F. Beckenbach, W. Seidel, and Otto Szász, 'Recurrent determinants of Legendre and ultraspherical polynomials', *Duke Math. Journal*, 18 (1951).
3. J. L. Burchnall and T. W. Chaundy, 'Expansions of Appell's double hypergeometric functions (II)', *Quart. J. of Math.* (Oxford), 12 (1941).
4. W. N. Bailey, *Generalized Hypergometric Series* (Cambridge, 1935).

A FURTHER NOTE ON TWO OF RAMANUJAN'S FORMULAE

By W. N. BAILEY (*London*)

[Received 10 September 1951]

1. In a recent note† I gave new proofs of Ramanujan's two formulae

$$\sum_{n=0}^{\infty} p(5n+4)x^n = 5\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{(1-x^n)^6}, \tag{1}$$

$$\prod_{n=1}^{\infty} \frac{(1-x^n)^5}{1-x^{5n}} = 1-5\left(\frac{x}{1-x}-\frac{2x^2}{1-x^2}-\frac{3x^3}{1-x^3}+\frac{4x^4}{1-x^4}+\frac{6x^6}{1-x^6}-\dots\right), \tag{2}$$

where $p(n)$ is the number of partitions of n. In the proof of (1) I gave Ramanujan's argument for deducing it from

$$x\prod_{n=1}^{\infty} \frac{(1-x^{5n})^5}{1-x^n} = \frac{x}{(1-x)^2}-\frac{x^2}{(1-x^2)^2}-\frac{x^3}{(1-x^3)^2}+\frac{x^4}{(1-x^4)^2}+\frac{x^6}{(1-x^6)^2}-\dots. \tag{3}$$

My proofs of (2) and (3) depended on the known sum of a well-poised bilateral basic series ${}_6\Psi_6$. The proof of (3) has now led me to proofs of (2) and (3) depending only on well-known formulae in elliptic functions which would be familiar to Ramanujan.

2. I assume the formula‡

$$\wp(u) = -\frac{\eta_1}{\omega_1}+\left(\frac{\pi}{2\omega_1}\right)^2 \operatorname{cosec}^2\frac{\pi u}{2\omega_1}-2\left(\frac{\pi}{\omega_1}\right)^2 \sum_{1}^{\infty} \frac{sq^{2s}}{1-q^{2s}}\cos\frac{s\pi u}{\omega_1},$$

and use the simple result

$$\sum_{-\infty}^{\infty}{}' \frac{aq^n}{(1-aq^n)^2} = \sum_{n=1}^{\infty}\left[\frac{aq^n}{(1-aq^n)^2}+\frac{q^n/a}{(1-q^n/a)^2}\right]$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m(a^m+a^{-m})q^{mn} = \sum_{m=1}^{\infty} \frac{m(a^m+a^{-m})q^m}{1-q^m}, \tag{4}$$

† 'A note on two of Ramanujan's formulae', above, 29–31.
‡ Tannery and Molk, *Fonctions elliptiques* (3), 118.

Quart. J. Math. Oxford (2), 3 (1952), 158–60.

where $|q| < |a| < 1/|q|$. If we write $a = \exp(\pi iu/\omega_1)$, we get

$$\sum_{-\infty}^{\infty} \frac{aq^{2n}}{(1-aq^{2n})^2} = -\frac{\eta\omega_1}{\pi^2} - \frac{\omega_1^2}{\pi^2}\wp(u).$$

Thus, if $b = \exp(\pi iv/\omega_1)$, we have

$$\sum_{-\infty}^{\infty} \left[\frac{aq^{2n}}{(1-aq^{2n})^2} - \frac{bq^{2n}}{(1-bq^{2n})^2}\right] = \frac{\omega_1^2}{\pi^2}[\wp(v)-\wp(u)] = \frac{\omega_1^2}{\pi^2}\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)}.$$

We therefore find, after a little reduction and changing q^2 into q, that

$$\sum_{-\infty}^{\infty} \left[\frac{aq^{n}}{(1-aq^{n})^2} - \frac{bq^{n}}{(1-bq^{n})^2}\right] = a\,\Pi\begin{bmatrix} ab, q/ab, b/a, qa/b, q, q, q, q; \\ a, a, q/a, q/a, b, b, q/b, q/b \end{bmatrix} \tag{5}$$

where $$\Pi\begin{bmatrix} a_1, a_2, \ldots; \\ b_1, b_2, \ldots \end{bmatrix} = \prod_{n=0}^{\infty} \frac{(1-a_1q^n)(1-a_2q^n)\ldots}{(1-b_1q^n)(1-b_2q^n)\ldots}.$$

The formula (5) is thus equivalent to the well-known expression for $\wp(u)-\wp(v)$ in terms of σ-functions. It is also equivalent to a particular case of the formula summing a ${}_6\Psi_6$.

3. For the proofs of both (2) and (3) we take $b = a^2$ in (5). First take $a = x$, $b = x^2$, $q = x^5$, and we get

$$\sum_{-\infty}^{\infty} \left[\frac{x^{5n+1}}{(1-x^{5n+1})^2} - \frac{x^{5n+2}}{(1-x^{5n+2})^2}\right] = x\prod_{1}^{\infty} \frac{(1-x^{5n})^5}{1-x^n},$$

which is (3).

Next take $a = \omega$, $b = \omega^2$, where $\omega = \exp(2\pi i/5)$, and the product on the right of (5) becomes

$$\frac{\omega(1-\omega^3)(1-\omega)}{(1-\omega)^2(1-\omega^2)^2}\prod_{1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}}.$$

The left-hand side of (5) is, by (4),

$$\frac{\omega}{(1-\omega)^2} - \frac{\omega^2}{(1-\omega^2)^2} + \sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}(\omega^m+\omega^{4m}-\omega^{2m}-\omega^{3m})$$
$$= \frac{\omega(1-\omega^3)(1-\omega)}{(1-\omega)^2(1-\omega^2)^2} + \sum_{m=1}^{\infty}\frac{mq^m\omega^m(1-\omega^m)(1-\omega^{2m})}{1-q^m}.$$

If we denote $\omega(1-\omega)(1-\omega^2)$ by A, it is easily seen that

$$\omega^m(1-\omega^m)(1-\omega^{2m})$$

has the values 0, A, $-A$, $-A$, A when m has the forms $5n$, $5n+1$, $5n+2$, $5n+3$, $5n+4$. Thus

$$\prod_{1}^{\infty}\frac{(1-q^n)^5}{1-q^{5n}} = 1+B\sum_{n=0}^{\infty}\left[\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}}-\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}-\right.$$
$$\left.-\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}+\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right],$$

where

$$\begin{aligned} B &= \omega(1-\omega)(1-\omega^2)\frac{(1-\omega)(1-\omega^2)^2}{\omega(1-\omega^3)} \\ &= -(1-\omega)(1-\omega^2)(1-\omega^3)(1-\omega^4) \\ &= -5. \end{aligned}$$

This completes the proof of (2). It is interesting to note that both (2) and (3) are merely particular cases of the formula expressing $\wp(u)-\wp(2u)$ in terms of σ-functions, when the $\wp$-functions are replaced by the corresponding Fourier series.

2,

th
ng
ed